Solution of Assignment - 10

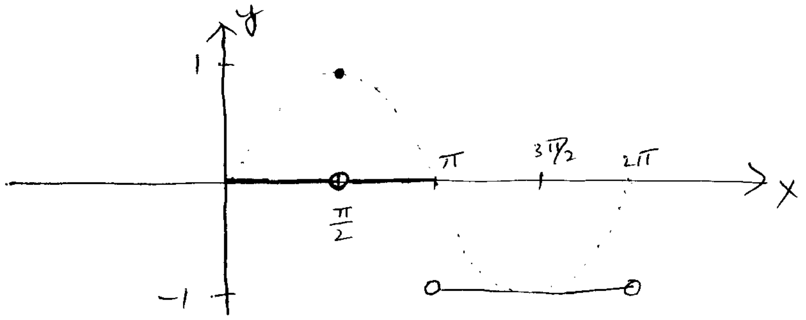
(1) (d) Right hand limit of $f(x)$ at $x = 0$

$$= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin\frac{1}{x} \text{ does not exist}$$

Illy LHL of $f(x)$ also does not exist.

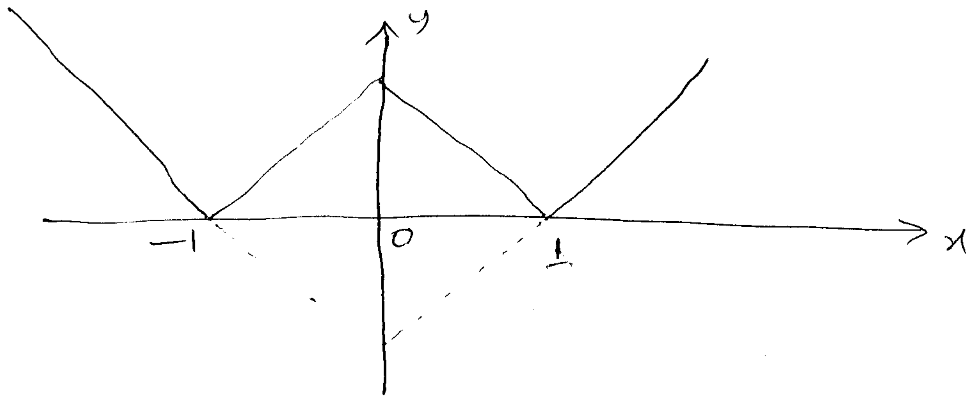
So, f has discontinuity of 2^{nd} kind at $x = 0$.

(2) (c) graph of $f(x) = [\sin x]$ is



Thus we see that f is continuous at $\frac{3\pi}{2}$

(3) (a) graph of $f(x) = ||x| - 1|$ is



Thus f is not differentiable at $-1, 0$ and 1

(4) (a)

$$g(x) = \begin{cases} k\sqrt{x+1} & 0 \le x \le 3 \\ mx+2 & 3 < x \le 5 \end{cases}$$

at $x = 3$, $RHL = 3m + 2$
$LHL = 2k$

$\therefore \; 2k = 3m + 2$ —— (1)

also,

$$g'(x) = \begin{cases} \dfrac{k}{2\sqrt{x+1}} & 0 \le x < 3 \\ m & 3 < x \le 5 \end{cases}$$

$L\{g'(3)\} = \dfrac{k}{4}$

and $R\{g'(3)\} = m$

Thus, $\dfrac{k}{4} = m$ —— (2)

(1) and (II) $\Rightarrow$

$k = \dfrac{8}{5}$ and $m = \dfrac{2}{5}$

$\therefore \; k + m = \dfrac{8}{5} + \dfrac{2}{5} = 2$

(5) (b)

$$f(x) = \frac{x}{1+|x|}$$

$$= \begin{cases} \dfrac{x}{1+x} & : x \geq 0 \\ \dfrac{x}{1-x} & x < 0 \end{cases}$$

clearly we see that the function is differentiable at $(-\infty, 0) \cup (0, \infty)$.
We need to check only at $x = 0$.

so, $$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{x}{1+|x|} - 0}{x}$$

$$= \lim_{x \to 0} \frac{1}{1+|x|} = 1$$

Hence, $f(x)$ is differentiable on $(-\infty, \infty)$.

(6) (b)

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h)}{h} - \lim_{h\to 0} \frac{f(1)}{h}$$

Since, $\lim_{h\to 0} \frac{f(1+h)}{h} = 5$

So, $\lim_{h\to 0} \frac{f(1)}{h}$ must be finite as $f'(1)$ exists and $\lim_{h\to 0} \frac{f(1)}{h}$ can be finite only if $f(1) = 0$ and $\lim_{h\to 0} \frac{f(1)}{h} = 0$.

$$\therefore f'(1) = \lim_{h\to 0} \frac{f(1+h)}{h} = 5.$$

(7) <u>(d)</u>

$f(1) = -2$ and $f'(x) \geq 2$

$$\Rightarrow \int_{f(1)}^{f(6)} f'(x)\,dx \geq \int_1^6 2\,dx$$

$$\Rightarrow f(x)\Big|_{f(1)}^{f(6)} \geq 2[x]_1^6$$

$$\Rightarrow f(6) - f(1) \geq 10$$

$$\Rightarrow f(6) \geq 10 + f(1)$$

$$\Rightarrow f(6) \geq 10 - 2 = 8$$

(8) (c)

$$f(x) = |x-1| + |x-3| + \sin x$$

By definition we can check that $f(x)$ is not differentiable at $x=1$ and $x=3$ and it is differentiable at all other points.

(9) (c)

$$f(x) = a\sin|x| + be^{|x|}$$

$$= \begin{cases} a\sin x + be^{x} & x \geq 0 \\ -a\sin x + be^{-x} & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \cancel{a\sin x} \; a\cos x + be^{x} & x > 0 \\ -a\cos x - be^{-x} & x < 0 \end{cases}$$

$LHL = b = RHL$

$Lf'(0) = -a-b$

$Rf'(0) = a+b$

$Lf'(0) = Rf'(0)$

$\Rightarrow -a-b = a+b$

$\Rightarrow 2(a+b) = 0$

$\Rightarrow a+b = 0$

(10) To check differentiability at $x=0$

$$R\{f'(0)\} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2\left|\cos\frac{\pi}{h}\right| - 0}{h}$$

$$= 0$$

|||ly $L\{f'(0)\} = 0$

so, $f(x)$ is diff. at $x=0$.

To check diff. at $x=2$

$$R\{f'(2)\} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2\left|\cos\left(\frac{\pi}{2+h}\right)\right| - 0}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2\cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2\sin\left(\frac{\pi}{2} - \frac{\pi}{2+h}\right)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2\sin\left(\frac{\pi h}{2(2+h)}\right)}{h\,\frac{\pi}{2(2+h)}} \cdot \frac{\pi}{2(2+h)}$$

$$= \pi$$

$$L\{f'(2)\} = \lim_{h\to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h\to 0} \frac{(2-h)^2 \left|\cos \frac{\pi}{2-h}\right| - 2^2 \left|\cos \frac{\pi}{2}\right|}{-h}$$

$$= \lim_{h\to 0} \frac{(2-h)^2 \cos\left(\frac{\pi}{2-h}\right) - 0}{-h}$$

$$= \lim_{h\to 0} \frac{-(2-h)^2 \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{2-h}\right)}{h}$$

$$= \lim_{h\to 0} \frac{(2-h)^2 \sin\left(\frac{\pi h}{2(2-h)}\right)}{h \times \frac{-\pi}{2(2-h)}} \times \frac{-\pi}{2(2-h)}$$

$$= -\pi$$

Thus, $f(x)$ is differentiable at $x = 0$
but not diff. at $x = 2$.