Solution of Assignment - 1

(1) We may write

\[ W = \{ b_k : b_k(x) = k \text{ for all } x \in \mathbb{R} \} \]

Now suppose a function

\[ f : W \rightarrow \mathbb{R} \text{ defined as} \]

\[ f(b_k) = k \]

then clearly

\[ f \text{ is a one-one onto, so,} \]

\[ |W| = |\mathbb{R}| \]

(2) To solve this question we will check each option one by one.

(a) \[ d(-2, -3) = \min(-2, -3) \]

\[ = -3 \leq 0 \]

so, it is not a metric.

(b) \[ d(x, y) = |x - y| \text{ satisfies all the properties to be a metric trivially.} \]

so, it is a metric on \( \mathbb{R} \).

(c) \[ d(-1, 1) = |(-1)^2 - 1^2| = 0 \]

but \(-1 \neq 1\)

so, it is not a metric.
\[ A = [0,1] \]

(a) \( \overline{A} = [0,1] \)

(b) \( \text{int}(A) = ]0,1[ \)

(c) \( \text{int}(A) = ]0,1[ \)

(d) \( A' = [0,1] \)

So, only (c) matches with the options given in the question.

(4) Trivial from definitions.

(5) Let \( S \) be infinite (follow lecture 1 for better explanation)

Consider \( F_n = \{ T \subseteq S : |T| = n \} \)

Since \( S \) is infinite, there exists subsets of all finite cardinalities.

So, \( F_n \) is non-empty.

Now in each \( F_n \), there must exist an element \( s_n \) which is not in other \( F_n \). Now the set collection of all these \( s_n \), say \( S' \), is such that \( S' \subseteq S \) and \( S' \) is countable.

So, every infinite set has a countable subset.
5) the option (b) of (5) is not always true. As for example.

\[ \mathbb{N} = \text{the set of natural no.} \]

is an infinite set but does not have an uncountable subset because it is itself a countable set.

(6)

\[
\begin{array}{c}
\text{A} \\
\text{B}
\end{array}
\]

Clearly, the picture of \( \mathcal{A} \) is a union of two convex sets. But the line \( AB \) does not contain in the union. So, union of convex sets can not be always convex.

Again, suppose \( S \) and \( T \) are two convex sets.

Let \( U = S \cup T \)

Let \( A \) and \( B \) be any arbitrary points of \( U \)

Let \( W = \{ x \mid x \text{ is a point of the line segment } AB \} \)
Now, \( A, B \in U = SAT \)
\[ \Rightarrow A, B \in S \text{ and } A, B \in T \text{ both} \]
\[ \Rightarrow W \in S \text{ and } W \subseteq T \text{ both} \]
\[ \text{since } S \text{ and } T \text{ both are convex} \]
\[ \Rightarrow W \subseteq SAT \]

So, \( SAT \) is convex.

(a) follow lecture 2

(b) suppose

\[ f : \mathbb{N} \rightarrow A \text{ s.t. } f(n) = \frac{1}{n} \]

Clearly, \( f \) is one-one and onto.

So, \( A \) is countable.

(c) \[ g : \mathbb{N} \rightarrow S \text{ by } g(n) = (0, n) \]

\[ \text{one-one onto} \]

So, \( S \) is also countable.

(d) Suppose,

\[ X = \{ x = (x_1, x_2, \ldots) \mid x_i = 0 \text{ or } 1 \} \]

It is already discussed in lecture (2) that \( X \) is uncountable.
Now, consider a mapping

\[ f : P(\mathbb{N}) \rightarrow X \text{ defined by} \]

\[ f(A) = (x_1, x_2, \ldots) \text{ such that} \]

\[ x_i = \begin{cases} 
1 & \text{if } i \in A \\
0 & \text{if } i \notin A 
\end{cases} \]

Clearly, \( f \) is one-one.

Consider any \( x = (x_1, x_2, \ldots) \in X \) then we can find a set \( B \),

\[ B = \{ i \mid x_i = 1 \} \subset \mathbb{N} \]

so, \( f \) is onto

\[ \therefore P(\mathbb{N}) \sim X \]

i.e. \( P(\mathbb{N}) \) is uncountable.
(8) (a) \[ d(2,3) = \max(2-3,0) \]
\[ = \max(-1,0) \]
\[ = 0 \]

But 2 \neq 3, so, \( d(z,d) \) is not a metric space.

(b) 1. \( d'(x,y) > 0 \) and \( d'(x,y) = 0 \) iff \( x = y \).
2. \( d'(x,y) = d'(y,x) \)
3. If \( d'(x,y) = 0 \), then
   Clearly, \( d'(x,y) \leq d'(x,z) + d'(z,y) \)
   \[ \text{if } \ d'(x,y) = 1, \text{ then } x \neq y \]

So, \( d'(x,z) + d'(z,y) = \begin{cases} 
2 & \text{if } z \neq x \neq y \\
1 & \text{if } z = \text{either } x \text{ or } y 
\end{cases} \)

In any case \( d'(x,y) \leq d'(x,z) + d'(z,y) \)

So, \( (x,d') \) is a metric space.

(c) \[ d_p(1,-1) = |1 - (-1)| = 0 \]

but, \( 1 \neq -1 \)

So, \( (\mathbb{Z},d_p) \) is not a metric space.
Both the points \((5, 4)\) and \((5, -4)\) are in the given set, but all the points of the line segment \((5, 4)\) joining \((5, 4)\) and \((5, -4)\) do not lie on the set because \(8\) is removed.

\[(b)\]

\[(c)\]
10 (a) union of open sets, so \( X \) is open.

(b) 
\[ y = x^3 \]
closed, because
\[ \text{int}(Y) = \emptyset \]
so \( \text{int}(Y) \neq Y \).
thus not open.

(c)
the annular ring
\( Z \setminus V \) contains
the inside boundary,
the points of which
are not interior
points of \( Z \setminus V \).
So, \( Z \setminus V \) is not
an open set.