Properties of measures (Part 1)

Let \((\mathbb{R}, \mathcal{F}, \mu)\) be a measure space.

In this lecture, we discuss certain algebraic properties of measures \(\mu\).

Note (1): From our discussion in the previous lectures, we have the following facts about measures.

(i) \(\mu(\emptyset) = 0\).

(ii) \(\mu\) is finitely additive and countably additive.

(iii) \(0 \leq \mu(A) \leq \mu(\mathbb{R}), \forall A \in \mathcal{F}\).

(iv) \(\mu(A) + \mu(A^c) = \mu(\mathbb{R}), \forall A \in \mathcal{F}\).

(v) Measures \(\mu\) with \(\mu(\mathbb{R}) = 1\) are said to be probability measures.

We need the following terminology to discuss further properties of measures.

**Definition** (Finite and countable disjoint unions)

(i) A collection \(\mathcal{E}\) of subsets of \(\mathbb{R}\) is said to be closed under finite disjoint unions, if for any positive integer \(n\) and pairwise disjoint sets
$A_1, A_2, \ldots, A_n \in \mathcal{E}$, we have $\bigcup_{i=1}^{n} A_i \in \mathcal{E}$.

(ii) A collection $\mathcal{E}$ of subsets of $\mathbb{N}$ is said to be closed under countable disjoint unions, if for any sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{E}$, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{E}$.

Using Note 10, we have the following result involving probability measures.

**Proposition 8:** Let $P_1$ and $P_2$ be two probability measures defined on the same measurable space $(\mathbb{R}, \mathcal{F})$. Then the collection

$$\mathcal{E} = \{ A \in \mathcal{F} \mid P_1(A) = P_2(A) \}$$

is non-empty and, closed under complementation and finite and countable disjoint unions.

**Proof:** Since $P_1(\mathbb{R}) = 1 = P_2(\mathbb{R})$, then $\mathbb{R} \in \mathcal{E}$.

Thus $\mathcal{E}$ is non-empty.

If $A \in \mathcal{E}$, then $P_1(A) = P_2(A)$. In this case,

$$P_1(A^c) = P_1(\mathbb{R}) - P_1(A) = P_2(\mathbb{R}) - P_2(A) = P_2(A^c),$$

hence $A^c \in \mathcal{E}$ and $\mathcal{E}$ is closed under
Complementation.

Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint sets in $\mathcal{E}$. Then $P_1(A_i) = P_2(A_i)$ for $i = 1, 2, \ldots, n$. By finite additivity of $P_1$ and $P_2$, we have,

$$P_1 \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} P_1(A_i) = \sum_{i=1}^{n} P_2(A_i) = P_2 \left( \bigcup_{i=1}^{n} A_i \right)$$

and hence $\bigcup_{i=1}^{n} A_i \in \mathcal{E}$, i.e. $\mathcal{E}$ is closed under finite disjoint unions.

The proof involving countable disjoint unions is similar and is left as an exercise.

We now discuss further properties of measures.

**Proposition 4**: For any $A, B \in \mathcal{E}$, we have

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

**Proof**: Note that, $A$ and $B$ can be written as a pairwise disjoint union of sets in the following form:

$$A = (A \cap B) \cup (A \cap B^c)$$

and $B = (A \cap \bar{B}) \cup (A^c \cap B)$. Then by finite additivity of $\mu$,
\[ \mu(A) + \mu(B) = \mu(AB) + \mu(\cap \setminus B) \setminus \mu(B) + \mu(\setminus B) \cap \mu(B) \cap \setminus B, \]

Again, by finite additivity of \( \mu \),

\[ \mu(A \cup B) = \mu(\cap \setminus B) + \mu(\setminus B) + \mu(\setminus B). \]

Hence, the equality follows.

**Corollary to Proposition 7**

For any \( A, B \in \mathcal{F} \) with \( B \subseteq A \),

we have

\( i) \mu(A) = \mu(B) + \mu(A \setminus B) \).

\( ii) \mu(B) \leq \mu(A) \).

**Proof:** Exercise.

**Note:** Given a sequence \( A_1, A_2, \ldots \) of sets in \( \mathcal{F} \), recall that by Exercise 2 of Week 1, we can construct sets \( B_1, B_2, \ldots \)
also in \( \mathcal{F} \) such that

\( i) \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \)

\( ii) \ B_n \subseteq A_n \ orall \ n \)

and \( iii) \ B_n \)'s are pairwise disjoint.

In fact, Exercise 2 was for any finite number of sets \( A_1, A_2, \ldots, A_n \). Here, we are repeating the construction of \( B_n \)'s for each \( n \) and thereby we obtain
a sequence \( \{B_n\}_n \) as above, for any given sequence \( \{A_n\}_n \).

**Proposition 8**: let \( \{A_n\}_n \) be a sequence in \( F \). Then \( \mu\left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu(A_n) \).

**Proof**: We use the sequence \( \{B_n\}_n \) as in the above note. Using properties of \( B_n \),

\[
\mu\left( \bigcup_{n=1}^{\infty} A_n \right) = \mu\left( \bigcup_{n=1}^{\infty} B_n \right)
\]

\[
= \sum_{n=1}^{\infty} \mu(B_n)
\]

\[
\leq \sum_{n=1}^{\infty} \mu(A_n).
\]

This completes the proof.

**Note 3**: For any finite measure, and in particular, for any probability measure, using Proposition 7, we have,

\[
\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)
\]

for any \( A, B \in F \). By the principle of Mathematical Induction, this result can be generalized to any finite number of sets. This is stated in the next result.

**Proposition 9**: (Inclusion-Exclusion Formula)

let \( \mu \) be a finite measure on \((\Omega, \mathcal{F})\).
and let $A_1, A_2, \ldots, A_n \in \mathcal{F}$. Then,

$$
\mu\left( \bigcup_{i=1}^{m} A_i \right) = \sum_{i=1}^{m} \mu(A_i) - \sum_{i,j=1}^{m} \mu(A_i \cap A_j) + \cdots + (-1)^{m-1} \mu(A_1 \cap A_2 \cap \cdots \cap A_n).
$$

**Proof:** Exercise.