Expectation of Absolutely Continuous RVs

In this lecture, we discuss about the computation of expectation for absolutely continuous random variables/ vectors. We shall also see some inequalities involving the moments of general RVs.

**Note 17:** Given an absolutely continuous RV \( X : (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) and a Borel measurable function \( g : (\mathbb{R}, \mathcal{B}_\mathbb{R}) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \), the RV \( g(X) : (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) is not necessarily absolutely continuous. For example, taking \( g \) to be a constant function, we get \( g(X) \) is a constant/degenerate RV.

To ensure that \( g(X) \) is absolutely continuous, we need additional regularity of \( g \). We recall a result,
usually discussed in basic probability courses. We state this without proof.

For brevity, we do not state a similar result for random vectors.

**Theorem 2:** Continue with the notations of Note 17. Let \( g: \mathbb{R} \to \mathbb{R} \) be differentiable with \( g(x) > 0 \ \forall x \in \mathbb{R} \). Then the RV \( Y = g(X) \) is also an absolutely continuous RV with p.d.f \( f_Y \) given by

\[
f_Y(y) = \begin{cases} 
  f_X(g'(y)) \frac{1}{g'(y)} & \text{if } y \in (g(-\infty), g(\infty)) \\
  0 & \text{otherwise},
\end{cases}
\]

where \( f_X \) is the p.d.f of \( X \), \( g(-\infty) := \lim_{x \to \infty} g(x) \) and \( g(\infty) := \lim_{x \to -\infty} g(x) \).

**Computation of expectation for absolutely continuous random variables/ vectors:**

let \( X: (\Omega, \mathcal{F}, P) \to (\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d}) \) be an absolutely continuous random vector.
with pdf $f_X$. By definition,

$$\frac{d \, P \circ x^{-1}}{d \, x^{(d)}} = f_X,$$

where $x^{(d)}$ is the Lebesgue measure on $\mathbb{R}^d$.

By Exercise 4(2),

$$EX = \int x(w) \, dP(w) = \int x \, dP \circ x^{-1}(x)$$

$$= \int \mathbb{R}^d x \cdot f_X(x) \, dx^{(d)}(x) = \int_{\mathbb{R}^d} x \cdot f_X(x) \, dx,$$

provided one of the integrals exist. Note that "$x" and "EX" in the above computation is $\mathbb{R}^d$-valued and the equality can be interpreted component-wise. Another way to interpret the above equalities is through the integration of $\mathbb{R}^d$-valued measurable functions with respect to measure on $\mathbb{R}^d$. This can be defined in a manner similar to the discussion for $\mathbb{R}$-valued functions in Week 6.

Note 18: Continue with the notations of Note 17 and write $Y = g \circ X = g(x)$. 
Then, show that (Exercise)

\[ \int_y dP_y(y) = EY = E g(x) = \int g(x) dP_x(x), \]

provided one of the integrals is defined.

By the above discussion,

\[ EY = E g(x) = \int g(x) f_x(x) dx, \]

provided the integral exists. Note that the expression is valid for any Borel measurable \( g \) and the RV \( Y \) is not necessarily absolutely continuous. By choosing appropriate functions \( g \), we consider the moments \( E(x-c)^n \) for \( X \).

Observe that the above expressions for expectation matches with those discussed in basic probability courses.

**Exercise 5:** Compute the moments of \( X \) when \( X \sim \text{Uniform}(0,1) \), \( \text{Exp}(\alpha) \) or \( N(\mu, \sigma^2) \).

**Note 19:** We may now repeat the usual analysis done in basic probability courses involving variance and
covariance etc.

Exercise 6: Check that $\mu = \mathbb{E} x$ and 
$\sigma^2 = \text{variance}(X) = \mathbb{E}(x - \mu)^2$ when $X \sim N(\mu, \sigma^2)$.

Note 20: Following the discussion in 
Note 26 of Week 7, we look at the 
characteristic functions of absolutely 
continuous RVs. For such RVs $x$,

$$\phi_x(u) = \mathbb{E} e^{iuX}, \quad u \in \mathbb{R}$$

$$= \int \cos((ux)) \, d\mathbb{P}^x \, \mathbb{R}$$

$$+ i \int \sin((ux)) \, d\mathbb{P}^x \, \mathbb{R}$$

$$= \int e^{iuX} \, f_x(x) \, dx, \quad \mathbb{R}$$

when $X$ is an $\mathbb{R}^d$-valued random vector, 
the characteristic function is defined in 
the following way: for $u \in \mathbb{R}^d$,

$$\phi_x(u) = \mathbb{E} e^{iu \cdot X}$$

$$= \mathbb{E} \exp\left( \sum_{j=1}^{d} i u_j X_j \right)$$

$$= \int e^{iu \cdot X(\omega)} \, d\mathbb{P}(\omega)$$

$\mathbb{R}$
\[ = \int e^{iu \cdot x} \, dP_{\text{law } x}(x). \]

As mentioned in Note 26 of Week 7, 
\( \phi_x \) uniquely determines the law of \( x \).

**Note 25**: Recall from Note 25 of Week 3 that \( x = (x_1, \ldots, x_d)^t \) is a random vector on a probability space \((\Omega, \mathcal{F}, P)\) if and only if \( x_j = 1, 2, \ldots, d \) are RVs on the same probability space. More generally, consider \( 1 \leq j_1 < j_2 < \cdots < j_n \leq d \) and look at the continuous map \( g: \mathbb{R}^d \to \mathbb{R}^n \) given by \( g(x_1, \ldots, x_d) := (x_{j_1}, \ldots, x_{j_n})^t \), \( \forall x \).

Then \( (x_{j_1}, \ldots, x_{j_n})^t = g(x) \) is an \( \mathbb{R}^n \)-valued random vector.

**Exercise 7**: Continue with the notations of Note 21. Show that \( (x_{j_1}, \ldots, x_{j_n})^t \) is discrete if \( (x_1, \ldots, x_d)^t \) is discrete. Can you make a similar statement for absolutely continuous random vectors?
Note 22: Continue with the notations of Note 21. As mentioned above, the law/distributions of \((X_{j_1}, \ldots, X_{j_n})^t\) for \(1 \leq n < d\) with \(1 \leq j_1 < \cdots < j_n \leq d\) can be obtained from the law/distribution of \(X = (X_1, \ldots, X_d)^t\). These \(n\)-dimensional distributions are referred to as the \(n\)-dimensional marginal distributions of the \(d\)-dimensional random vector \(X\).

Given \(X\), the marginal distributions are uniquely determined. However, the converse is not true.

Exercise 8: Find an example showing that the marginal distributions do not uniquely determine the distribution of a random vector.