Riemann and Lebesgue integration

In this lecture, we compare the Riemann integration on $\mathbb{R}$ and the integration with respect to the Lebesgue measure $\lambda$ on $\mathbb{R}$. The second type of integration shall be referred to as the Lebesgue integration.

Note (14): In this lecture, we denote the Riemann integrals by $(R) \int \ldots \, dx$. The lebesgue integrals shall be denoted by $\int \ldots \, \lambda(dx)$ or $\int \ldots \, d\lambda$.

Note (15): (i) Given $f : (\mathbb{R}, \mathcal{B}_\mathbb{R}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ measurable and any bounded interval $[a, b]$, we may consider the integral $\int f \, d\lambda = \int f \, 1_{[a,b]} \, dx$. Since $\lambda([a,b]) = \lambda([b,a]) = 0$, by Note (4),

$$\int f \, d\lambda = \int f \, dx = \int f \, dx = \int f \, d\lambda,$$

provided any one of the integrals exist.

(ii) Continue with $[a, b]$ as above.
using the idea of restrictions of the Lebesgue measure on Borel subsets in Note 29 of Week 5, we may consider the integral \[ \int_{[a,b]} g \, d\lambda \] for any \( g : [a,b] \rightarrow [a,b] \):

\( ([a,b], \mathcal{B}_{[a,b]}) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) measurable. We shall write \( \int g \, d\lambda \) here to simplify the notation.

**Exercise 3:** Take \( f \) and \([a,b]\) as in Note 15(i). Show that

(i) \( f \mid_{[a,b]} : ([a,b], \mathcal{B}_{[a,b]}) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) is measurable.

(ii) \( \int_{[a,b]} f \, d\lambda = \int f \, d\lambda \), if any one of the integrals exist.

**Note 16:** Even though \( (\mathbb{R}) \int \ldots \, dx \) and \( \int \ldots \, d\lambda \) both aim to compute the “area under the curve”, there are differences. We first do a preliminary comparison by taking various functions \( f : \mathbb{R} \rightarrow \mathbb{R} \).

(i) Take \( f = 1_{[0,1]} \), i.e.
\[
f(x) = \begin{cases} 
1, & \text{if } x \in [0,1] \\
0, & \text{otherwise.}
\end{cases}
\]

Then,

\[
\left(\text{R}\right) \int_{-\infty}^{\infty} f(x) \, dx = \left(\text{R}\right) \int_{0}^{1} f(x) \, dx = 1.
\]

\[
\left(\text{L}\right) \int \chi_{\mathbb{R}}(x) \, dx = \int_{\mathbb{R}} 1_{[0,1]}(x) \, dx
\]

\[= \lambda([0,1]) = 1 - 0 = 1.\]

Again,

\[
\left(\text{R}\right) \int_{0}^{1} f(x) \, dx = \left(\text{R}\right) \int_{0}^{1} 1 \, dx = 1,
\]

\[
\left(\text{L}\right) \int f \, dx = \int \chi_{[0,1]}(x) \, dx
\]

\[= \int 1_{[0,1]}(x) \, dx = 1.\]

Here, the integrals match.

(ii) Let \( Q \) denote the set of rational numbers. Take

\[
f(x) = \begin{cases} 
1, & \text{if } x \in [0,1] \cap Q \\
0, & \text{otherwise}
\end{cases}
\]

\[= 1_{[0,1] \cap Q}(x).
\]

Then, \( f \) is discontinuous on \([0,1]\) and is not Riemann integrable (Exercise).

But, \( \int_{\mathbb{R}} f \, dx = \lambda([0,1] \cap Q) = 0 \), since \([0,1] \cap Q\) is a countable set. As \( f \) is
non-negative, \( f \) is Lebesgue integrable.

(iii) For \( n = 1, 2, \ldots \), consider the set
\[
\mathcal{Q}_n = \left\{ \frac{p}{q} \in \mathbb{Q} \mid q = 1, 2, \ldots, n \right\}
\]
with \( p = -q, -(q-1), \ldots, -1, 0, 1, \ldots, (q-1), q \).

Note that \( \mathcal{Q}_n \) is finite for each \( n \) and \( \mathcal{Q}_n \uparrow \mathbb{Q} \). Then \( \left[0,1\right] \cap \mathcal{Q}_n \uparrow \left[0,1\right] \cap \mathbb{Q} \). Here,
\[
f_n := \frac{1}{\left[0,1\right] \cap \mathcal{Q}_n} \uparrow f := \frac{1}{\left[0,1\right] \cap \mathbb{Q}}.
\]
Check that \( f_n \)'s are both Riemann and Lebesgue integrable (Exercise). But, as observed above the limit function is Lebesgue integrable, but not Riemann integrable.

**Theorem 5:** Let \( f : [a, b] \to \mathbb{R} \) be bounded.

(i) If \( f \) is Riemann integrable, then \( f \) is Lebesgue integrable and
\[
\int_a^b f(x) \, dx = \int_{[a, b]} f(x) \chi(dx) \tag{R}
\]
(ii) \( f \) is Riemann integrable if and only if the set of discontinuities of \( f \), i.e.,
\[
\{ x \in [a, b] \mid f \text{ is discontinuous at } x \}
\]
has Lebesgue measure zero. Equivalently, we may state \( f \) is continuous \( \lambda \)-a.e.

We shall discuss the proof of Theorem 5 in the next lecture. In particular, the measurability of \( f \) in part (i) needs justification. We now focus on the interpretation of this result.

**Note 17:** Keeping the part (i) of Theorem 5 in mind, we now write \( \int_a^b f(x) \, dx \) for\(\int_a^b f(x) \, dx \), provided the integral makes sense. We treat \( \int_a^b f(x) \, dx \) as an extension of Riemann integration. Even though all Riemann integrable functions are Lebesgue integrable, the converse is not true (see Note 16 above).

**Note 18:** We now discuss the connection between \( \int_{-\infty}^{\infty} f(x) \, dx \) and \( \int_{-\infty}^{\infty} f(x) \, A(dx) \), \( A \) R

If \( f \) is bounded and Riemann integrable, then by Theorem 5(i) above,
\[ (R) \int_{-\infty}^{\infty} f(x) \, dx = \lim_{n \to \infty} \int_{-n}^{n} f(x) \, dx \]

\[ = \lim_{n \to \infty} \int_{-n}^{n} f(x) \, \lambda(dx) \]

\[ = \int_{\mathbb{R}} f(x) \, \lambda(dx), \]

provided the limits exist. We shall now write \( \int_{-\infty}^{\infty} f(x) \, dx \) for \( \int_{\mathbb{R}} f(x) \, \lambda(dx) \).

\textbf{Note 19:} If a function \( f \) has a singularity at any interior/boundary point of the interval and provided the “Riemann integral” can be defined, then we can consider the connection with Lebesgue integrals. To reduce technical details, we shall assume this equality in this case.

\textbf{Note 20: (A pictorial comparison)}

For simplicity, we work with bounded and non-negative \( f: [a,b] \to \mathbb{R} \). If \( f \) is Riemann integrable, then the area under the curve \( (R) \int_{a}^{b} f(x) \, dx \) is approximated by “vertically splitting” the
domain as follows

For the Lebesgue integral \( \int f \, dx \), we approximate \( f \) by simple functions from below and then by an application of the MCT, we obtain the approximation for the integral. This leads to "splitting the area" in "horizontal stripes" as follows.