Monotone Convergence Theorem

We have discussed some basic properties of measure theoretic integration in the previous lectures. We now state an important result involving this integration. The proof shall be discussed in Week 7.

Theorem 1: (Monotone Convergence Theorem)

Let \( \{h_n\} \) be a non-decreasing sequence of non-negative, Borel measurable functions defined on a measure space \((\Omega, \mathcal{F}, \mu)\). Suppose that the limit function \( h := \lim_{n \to \infty} h_n \) is real valued. Then

\[ Sh_n \uparrow S h \mu \text{ as } n \to \infty. \]

Note 17: (i) We shall use the term MCT to refer to the Monotone Convergence Theorem.

(ii) In the MCT, \( h \) is measurable since it is the pointwise limit of measurable functions.

(iii) In the case that \( h \) takes
values in $\overline{R}$, MCT continues to hold. We need to interpret $Sh\mu$ as an integral of $\overline{R}$-valued measurable function.

(iv) Given any non-negative and Borel measurable function, by Theorem 4 of Week 3, we can construct a sequence $Shn$ of simple functions with $0 \leq Shn \uparrow h$. This sequence falls in the setup of MCT.

(v) Given any non-negative and Borel measurable function $h$, using (iv) above and MCT, we have

$$Sh\mu = \lim_{n \to \infty} Shn\mu.$$ 

If we already have a definition of $Shn\mu$ for the simple functions $hn$, then $Sh\mu$ may be computed as above. This gives us an alternative way to compute $Sh\mu$ via limits, instead of supremum as mentioned in the definition.

(vi) In the setup of the MCT, we shall write $hn \uparrow h$. This is extending the usage of the notation "$\uparrow$" used for simple functions earlier.
(VIII) Given \( h = \lim_{n \to \infty} h_n \), the MCT gives sufficient conditions on the \( h_n \)’s which allows us to write

\[
\int (\lim_{n \to \infty} h_n) \, d\mu = \lim_{n \to \infty} \int h_n \, d\mu.
\]

In later lectures, we shall study more results of this type.

We now look at various important applications of the MCT.

**Theorem 2.** Let \( f, g : (\Omega, \mathcal{F}, \mu) \to (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) be measurable functions. If \( \int f \, d\mu \) and \( \int g \, d\mu \) exist and if \( \int f \, d\mu + \int g \, d\mu \) can be defined (i.e. “\( \infty - \infty \)” situation does not occur), then \( \int (f+g) \, d\mu \) exists and

\[
\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu.
\]

**Note 18:** We shall discuss the proofs of Theorems 1 and 2 in Week 7.

The next result is a consequence of Theorem 2.

**Corollary 1:** Let \( f \) be integrable. Then
If $f$ is also integrable (i.e. $\int f^+\,d\mu = \int f^-\,d\mu < \infty$; here $f^+ \equiv 0$, as $f$ is non-negative).

**Proof:** Since $f$ is integrable, we have

$$0 \leq \int f^+\,d\mu < \infty, \quad 0 \leq \int f^-\,d\mu < \infty.$$  

But $|f| = f^+ + f^-$ and by Theorem 2

$$\int |f|\,d\mu = \int (f^+ + f^-)\,d\mu = \int f^+\,d\mu + \int f^-\,d\mu < \infty.$$  

This completes the proof.

**Note 19:** Corollary 1 states the converse of Note 9. Combining both results, we have

$$f \text{ is integrable} \iff \int |f|\,d\mu < \infty.$$  

In this case, $\int f\,d\mu \in \mathbb{R}$.

**Note 20:** For an RV $X: (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$,

(i) $X$ is integrable $\iff \int |X|\,d\mathbb{P} < \infty \iff \mathbb{E}|X| < \infty$.  

In this case, $\mathbb{E}X = \int X\,d\mathbb{P} \in \mathbb{R}$.

(ii) $X$ is quasi-integrable

$\Rightarrow$ one of $\int X^+\,d\mathbb{P}$ and $\int X^-\,d\mathbb{P}$ is finite and the other is $+\infty$.

$\Rightarrow \mathbb{E}|X| = \int |X|\,d\mathbb{P} = \int X^+\,d\mathbb{P} + \int X^-\,d\mathbb{P} = +\infty.$
In this case, $\mathbb{E}x = \pm \infty$.

**Note 21:** In basic probability courses, for the existence of $\mathbb{E}x$, we consider the condition $\mathbb{E}|x| < \infty$. In addition, to this case, from now onwards we may still talk about $\mathbb{E}x$ if $x$ is quasi-integrable.

**Note 22:** For RVs $x, y : (\Omega, \mathcal{F}, \mu) \rightarrow (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$, under appropriate hypothesis, Theorem 2 implies $\mathbb{E}(x+y) = \mathbb{E}x + \mathbb{E}y$.

**Note 23:** Combining Theorem 2 and Proposition 1(i), we have the linearity of the map $h \mapsto \mathbb{E}h\mu$.

**Corollary 2:** If $f, g : (\Omega, \mathcal{F}, \mu) \rightarrow (\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ are integrable, then so is $f+g$.

**Proof:** We need to check $\int |f+g| d\mu < \infty$.

Since $f$ and $g$ are integrable, $\int |f| d\mu < \infty$ and $\int |g| d\mu < \infty$. By Proposition 1(ii),

$\int |f+g| d\mu \leq \int |f| d\mu + \int |g| d\mu < \infty$. Here, we use the inequality $|f+g| \leq |f| + |g|$.
Corollary 3: Let $\{h_n\}_n$ be a sequence of non-negative Borel measurable functions. Then
$$\int \left( \sum_{n=1}^{\infty} h_n \right) \, d\mu = \sum_{n=1}^{\infty} \int h_n \, d\mu.$$ 

Proof: we have $\sum_{n=1}^{m} h_n \uparrow \sum_{n=1}^{\infty} h_n$ and by

Theorem 2, $\int \left( \sum_{n=1}^{m} h_n \right) \, d\mu = \sum_{n=1}^{m} \int h_n \, d\mu$. 

Applying MCT, we have the result.