Properties of Measure Theoretic Integration

(Part 1)

In the previous lecture we have defined the integral \( \int h(d\mu) = \int h(\omega) d\mu(\omega) \)
\[ \int h(\omega) \mu(d\omega), \text{ where } h: (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \text{ is measurable and } \mu \text{ is a measure on } (\Omega, \mathcal{F}). \]

Note 7: The integral \( \int h \, d\mu \), as defined in the previous lecture, is not easily computable. We shall study properties of the integral and find easier ways to compute it.

Proposition 1: Let \( g, h : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) be measurable such that \( \int g \, d\mu \) and \( \int h \, d\mu \) exists (i.e. \( g \) is integrable or quasi-integrable. A similar statement holds for \( h \)).

(i) Fix \( c \in \mathbb{R} \). Then \( \int c h \, d\mu \) exists and \( \int c h \, d\mu = c \int h \, d\mu \).

Proof: We divide the argument into the following cases.

Case 1: \( c = 0 \).

Then \( ch = 0 \cdot 1_\Omega \) and hence \( \int ch \, d\mu = 0 \).
Again $c \int h \, d\mu = 0$. This proves the equality.

**Case 2** $c > 0$.

If $h = 1_A$ for some $A \in \mathcal{F}$, then
\[ \int ch \, d\mu = \int c \cdot 1_A \, d\mu = c \mu(A) = c \int h \, d\mu. \]

If $h$ is a simple function, then a computation similar to above yields the required equality.

If $h$ is a non-negative measurable function, then
\[ \int ch \, d\mu = \sup \left\{ \int s \, d\mu \mid 0 \leq s \leq ch, s \text{ is simple} \right\} \]
\[ = \sup \left\{ c \int \frac{s}{c} \, d\mu \mid 0 \leq \frac{s}{c} \leq h, \frac{s}{c} \text{ is simple} \right\} \]
\[ = \sup \left\{ c \int \frac{s}{c} \, d\mu \mid 0 \leq \frac{s}{c} \leq h, \frac{s}{c} \text{ is simple} \right\} \]
\[ = c \sup \left\{ \int \frac{s}{c} \, d\mu \mid 0 \leq \frac{s}{c} \leq h, \frac{s}{c} \text{ is simple} \right\} \]
\[ = c \int h \, d\mu. \]

If $h$ is $R$-valued measurable, then observe that $(ch)^+ = ch^+$ and $(ch)^- = ch^{-}$.

Therefore,
\[ \int ch \, d\mu = \int (ch)^+ \, d\mu - \int (ch)^- \, d\mu. \]
\[ = S \chi^+ d\mu - S \chi^- d\mu \]
\[ = c \left( S \chi^+ d\mu - S \chi^- d\mu \right) = c S \chi d\mu. \]

Case 3 \( c < 0 \).

In this case, we can write
\[ \chi = (-c)(-h) \]
with \( -c > 0 \) and \( -h \) being Borel measurable.
Moreover, \((\chi h)^+ = ((-c)(-h))^+ = (c)(h)^+ = (c)h^+\)
and \((\chi h)^- = ((-c)(-h))^- = (c)(h)^- = (c)h^-\).

The rest of the argument is similar to Case 2.

(ii) If \( h \leq g \) (i.e., \( h(w) \leq g(w) + \omega \leq r \)), then
\[ S \chi d\mu \leq S g d\mu. \]

Proof: First consider the case \( 0 \leq h \leq g \).

For any simple function \( \delta \) satisfying \( 0 \leq \delta \leq h \), we have \( 0 \leq \delta \leq g \). Hence,
\[ \{ S \delta d\mu \mid 0 \leq \delta \leq h \} \subseteq \{ S \delta d\mu \mid 0 \leq \delta \leq g \}. \]

The required inequality follows by taking supremum on both sides.

For the other case, \( h \) may now take both positive and negative values. The proof is similar to the case above, by carefully accounting for \( h^+ \) and \( h^- \).
explicit calculation is being skipped to reduce technicalities in the notes. The interested reader may refer to Theorem 1.5.9 from "Probability and Measure Theory" by Robert B. Ash and Catherine A. Doleans-Dade, Second edition, Academic Press.

(iii) \(|Sh\, dl| \leq S|hl| \, dl\).

Proof: Since \(|hl|\) is non-negative and measurable, \(\int |hl| \, dl\) exists and \(\int |hl| \, dl \geq 0\).

Now, \(-|hl| \leq h \leq |hl|\) and hence

\[-\int |hl| \, dl \leq \int h \, dl \leq \int |hl| \, dl.\]

The required inequality follows.

Note (8): If \(h\) is non-negative and integrable, then \(h = h^+\), \(h^- = 0\) and \(\int h^+ \, dl \leq \infty\).

If, in addition, \(|g| \leq h\), then \(0 \leq g^+ \leq |g| \leq h\) and \(0 \leq g^- \leq |g| \leq h\). Thus,

\[0 \leq \int g^+ \, dl < \infty\] and \(0 \leq \int g^- \, dl < \infty\).

Hence, \(g\) and \(|g|\) are integrable in this case.
Note 9: If $|g|$ is integrable, then by Note 8, $g$ is integrable.

Note 10: We are yet to discuss the linearity property $\int (g + h) \, d\mu = \int g \, d\mu + \int h \, d\mu$. As such, before proving a statement of this form, we cannot use this in our argument.

Note 11: Once we prove the statement in Note 10, we shall prove the converse of the statement in Note 9.