Distribution Functions
and Probability Measures
in higher dimensions

In the last few lectures, we have discussed the correspondence between distribution functions in \( \mathbb{R} \) and probability measures on \( (\mathbb{R}, \mathcal{B}_\mathbb{R}) \). We also saw that this correspondence is a special case of the correspondence between Lebesgue–Stieltjes measures on \( (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) and non-decreasing right-continuous functions on \( \mathbb{R} \). As an example of this correspondence, we found the Lebesgue measure on \( \mathbb{R} \) corresponding to the function \( F: \mathbb{R} \to \mathbb{R} \), defined by \( F(x) := x, \forall x \in \mathbb{R} \).

In this lecture, we discuss the analogues of these results in \( \mathbb{R}^2 \). The extension to higher-dimensional
Spaces $\mathbb{R}^d$, $d \geq 3$ can be done in an analogous fashion.

**Note 27:** Recall from week 1 that the Borel $\sigma$-field $\mathcal{B}_{\mathbb{R}^d}$ on $\mathbb{R}^d$ is taken as $\mathcal{B}_{\mathbb{R}^d} = \sigma \left( \prod_{i=1}^d (a_i, b_i] \mid -\infty < a_i < b_i < \infty \right)_{i=1, 2, \ldots, d}$.

**Note 28:** Definition 2 of Lebesgue-Stieltjes measures used bounded intervals. The analogue of bounded intervals in higher dimensions are the sets $\prod_{i=1}^d (a_i, b_i]$ with $-\infty < a_i < b_i < \infty$, $i = 1, 2, \ldots, d$.

**Definition 3 (Lebesgue-Stieltjes measure on $\mathbb{R}^d$):** A measure $\mu$ defined on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ is said to be a Lebesgue-Stieltjes measure if $\mu \left( \prod_{i=1}^d (a_i, b_i] \right) < \infty$ for all $-\infty < a_i < b_i < \infty$ for $i = 1, 2, \ldots, d$.

**Note 29:** Any finite measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ is Lebesgue-Stieltjes.

**Note 30:** Now, consider the analogue of
the notions \textit{“non-decreasing”} and \textit{“right-continuous”} in higher dimensions.

\textbf{(i)} Say that $F : \mathbb{R}^d \rightarrow \mathbb{R}$ is right-continuous if it is jointly right-continuous in all the variables, i.e.

$$\lim_{\mathbf{z}_i \downarrow \mathbf{x}_i} F(z_1^{(m)}, \ldots, z_d^{(m)}) = F(x_1, \ldots, x_d).$$

This is the same condition that we saw in Note 9 of Week 4 for the distribution function $F$ of an $\mathbb{R}^2$-valued random vector.

\textbf{(ii)} Taking motivation from Note 9 of Week 4, say that $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is non-decreasing if

$$F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1) \geq 0$$

for all $a_1 < a_2$, $b_1 < b_2$. We mention the version for higher-dimensions in the next exercise.

\underline{Exercise 5}: (i) Let $\mu$ be a finite measure
on $\mathbb{R}^2$. Consider the function $F_\mu : \mathbb{R}^2 \to \mathbb{R}$ defined by $F_\mu(x,y) := \mu((\infty, x] \times (\infty, y])$, for all $(x,y)^t \in \mathbb{R}^2$. Check that $F_\mu$ is non-decreasing and right-continuous.

(ii) let $\mu$ be a finite measure on $\mathbb{R}^d$. Consider the function $F_\mu : \mathbb{R}^d \to \mathbb{R}$ defined by $F_\mu(x_1, \ldots, x_d) := \mu(\prod_{i=1}^{d} (-\infty, x_i])$, for all $(x_1, \ldots, x_d)^t \in \mathbb{R}^d$. Check that $F_\mu$ is right-continuous. Write down the quantity $\mu(\prod_{i=1}^{d} [a_i, b_i])$ for $-\infty < a_i < b_i < \infty$ $i = 1, 2, \ldots, d$ in terms of the values $F_\mu(x)$.

Use the fact

$$\mu(\prod_{i=1}^{d} (a_i, b_i]) \geq 0$$

to get the non-decreasing property of $F_\mu$.

(iii) Continue with $F_\mu$ as in (ii). Fix $i \in \{1, 2, \ldots, d\}$. Then, show that for every fixed $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d$ in $\mathbb{R}$, the function

$$x \mapsto F_\mu(x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_d)$$
is non-decreasing.

Note 5: If we consider $\mu$ to be an infinite Lebesgue-Stieltjes measure in Exercise 5, then we need to consider $F_\mu: \mathbb{R}^d \to \mathbb{R}$. To avoid notational complexity, we have avoided this case.

The construction of a Lebesgue-Stieltjes measure on $\mathbb{R}^2$ from non-decreasing, right-continuous functions on $\mathbb{R}^2$ is stated in the next result. The idea of the proof remains the same as considered in the one-dimensional case and we do not repeat the arguments for brevity. This result also completes the correspondence between the Lebesgue-Stieltjes measures and the non-decreasing, right-continuous functions on $\mathbb{R}^2$. 
Theorem 0: Let $F: \mathbb{R}^2 \to \mathbb{R}$ be non-decreasing and right-continuous. Consider a set function $\mu$ given by

$$
\mu \left( \prod_{i=1}^{2} (a_i, b_i] \right) := F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1)
$$

$$
\iff -\infty < a_i < b_i < \infty, \ i = 1,2
$$

has a unique extension to a Lebesgue-Stieltjes measure on $\mathbb{R}^2$.

Note 32: If $F: \mathbb{R}^2 \to \mathbb{R}$ is non-decreasing and right-continuous with

(i) $\lim_{x \to \infty} F(x, y) = 1$

(ii) $\lim_{x \to -\infty} F(x, y) = 0 \forall y \in \mathbb{R}$

(iii) $\lim_{y \to -\infty} F(x, y) = 0 \forall x \in \mathbb{R}$

then the Lebesgue-Stieltjes measure obtained from Theorem 0 will be a probability measure. This also completes the identification between probability measures and appropriately defined "distribution functions" in $\mathbb{R}^2$. 
Note 33: Theorem 1 and Note 32 have appropriate extensions to higher dimensional Euclidean spaces $\mathbb{R}^d$, $d \geq 3$.

Note 34: (Lebesgue measure on $\mathbb{R}^d$)

The Lebesgue–Stieltjes measure $\lambda^{(d)}$ on $\mathbb{R}^d$ corresponding to the function $F: \mathbb{R}^d \to \mathbb{R}$ defined by $F(x_1, \ldots, x_d) := x_1 x_2 \ldots x_d$ is called the Lebesgue measure on $\mathbb{R}^d$.

Here, $\lambda^{(d)} \left( \prod_{i=1}^d (a_i, b_i] \right) = \prod_{i=1}^d (b_i - a_i)$, which matches the usual "area/volume" in $\mathbb{R}^d$. $\lambda^{(d)}$ has properties similar to $\lambda^1$, the Lebesgue measure on $\mathbb{R}$, such as singletons have zero size/mass.