Properties of Lebesgue measure on $\mathbb{R}$

In the previous lecture, we discussed the construction of a class of measures on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$, called the Lebesgue-Stieltjes measures, corresponding to the class of non-decreasing right-continuous functions $F: \mathbb{R} \to \mathbb{R}$. This construction generalized the correspondence between probability measures on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ and distribution functions $F: \mathbb{R} \to [0,1]$.

As a special case of this class of Lebesgue-Stieltjes measures, we have mentioned the example of the Lebesgue measure on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$. This measure appears in relation with the non-decreasing and right-continuous function $F(x) = x$, for $x \in \mathbb{R}$.

In this lecture, we discuss various
properties of this measure.

**Note 19:** We use $\lambda$ to denote the Lebesgue measure on $(\mathbb{R},\mathcal{B}_\mathbb{R})$. As mentioned in the previous lecture,

$$\lambda((a,b]) = b - a, \quad -\infty < a < b \leq \infty.$$ Therefore, $\lambda$ associates the usual "length" as the size of the intervals.

**Note 20:** For any $x \in \mathbb{R}$, consider the intervals $(x - \frac{1}{n}, x]$, $n = 1, 2, \ldots$. Since $\lambda((x-1, x]) = 1 < \infty$, we can apply the continuity from above (see Proposition 10 of Week 2). Now, $(x - \frac{1}{n}, x] \downarrow \{x\}$ and hence,

$$\lambda(\{x\}) = \lim_{n \to \infty} \lambda((x - \frac{1}{n}, x])$$

$$= \lim_{n \to \infty} \frac{1}{n} = 0.$$ 

**Note 21:** Since any finite or countably infinite set is a finite or countable disjoint union of singleton sets, by the finite/countable additivity of $\lambda$, we have $\lambda(A) = 0$ for any such set $A$. 
Note 22: For \( a \leq b \),
\[ \lambda([a,b]) = \lambda([a,b]) + \lambda([a,\infty)) = \lambda([a,b]) \]
Similar arguments will show the following equality
\[ \lambda([a,b]) = \lambda([a,b]) = \lambda((a,b)) \]
\[ = \lambda([a,b]) = b - a. \]

Note 23: There are uncountable sets \( A \subseteq \mathbb{R} \) with \( \lambda(A) = 0 \). One such example is the Cantor set. In this course, we do not go into the details about such sets.

Note 24: Recall the definitions of Borel \( \sigma \)-fields on Borel subsets of \( \mathbb{R} \). In particular, the Borel \( \sigma \)-field \( \mathcal{B}_{[0,1]} \) on \([0,1]\) consists of sets of the form \( A \cap [0,1] \), \( A \in \mathcal{B}_\mathbb{R} \). Consider \( \lambda : \mathcal{B}_{[0,1]} \to [0,\infty) \) defined by
\[ \lambda(A \cap [0,1]) := \lambda(A \cap [0,1]) \quad \forall A \in \mathcal{B}_\mathbb{R}. \]
Check that $\lambda$ is a measure
on $B_{(0,1)}$. (Exercise)

Moreover, $\lambda_{|_{(0,1)}} = \lambda((0,1)) = 1$.

Thus, $\lambda_{|_{B_{(0,1)}}}$ is a probability
measure on $(\mathbb{R}, B_{(0,1)})$. This
statement can be repeated for
$[a, b]$ with $b-a = 1$. We can
think of such measures as the
restriction of the Lebesgue measure
to the Borel subsets.

Note 25: (A subset of $\mathbb{R}$ which is
not in the collection $B_{\mathbb{R}}$)

Using set theoretic arguments,
specifically the axiom of choice and
equivalence relations, it is possible to
construct a sequence $\{A_n\}_{n=1}^\infty$ of
pairwise disjoint sets such that
$\bigcup_{n=1}^\infty A_n = (0,1)$. The sets can be chosen
Such that if \( \forall n \in \mathbb{N} \), then
\[
\lambda(A) = \lambda(A_n) \quad \forall n.
\]

In this case,
\[
1 = \lambda([0,1]) = \sum_{n=1}^{\infty} \lambda(A_n) = \sum_{n=1}^{\infty} \lambda(A_n)
\]

which leads to a contradiction. Thus \( A_n \notin \mathbb{B}_R \). To avoid the complexity of the construction of such sets, in this course we do not go into the details. The interested reader is referred to the book "Real Analysis, Modern Techniques and Their Applications" by Gerald Folland, second edition, John Wiley & Sons. (Chapter 1)

**Exercise 4:** For \( x \in \mathbb{R} \), consider the continuous function \( f : \mathbb{R} \to \mathbb{R} \) defined by
\[
f(y) := y - x \quad \forall y \in \mathbb{R}.
\]

(i) For any \( A \in \mathbb{B}_R \), check that
\[
A + x = f^{-1}(A)
\]

where \( A + x \) is defined as
\[
A + x := \{ y + x \mid y \in A \}.
\]
(ii) Show that \( A + x \in \mathcal{B}_R \) for all \( x \in \mathbb{R} \) and \( A \in \mathcal{B}_R \).

(iii) Show that the set function \( \mu : \mathcal{B}_R \to [0, \infty) \) defined by

\[
\mu(A) := \lambda(A + x), \quad \forall A \in \mathcal{B}_R
\]

is a \( \sigma \)-finite measure.

(iv) Check that \( \mu(A) = \lambda(A) \), for all \( A \) in the field of finite disjoint union of left-open right-closed intervals in \( \mathbb{R} \).

(v) Show that \( \mu(A) = \lambda(A) \), \( \forall A \in \mathcal{B}_R \).

Note (26): Exercise (4) implies that

\[
\lambda(A + x) = \lambda(A), \quad \forall x \in \mathbb{R}, \quad A \in \mathcal{B}_R.
\]

This is stated as follows: The Lebesgue measure \( \lambda \) is translation invariant.