Carathéodory Extension Theorem

In this lecture, we discuss extensions of measures. Recall the definitions of $\sigma$-finite measures on $\sigma$-fields and of measures on fields from Week 2. Given a non-negative finitely/countably additive set function $\mu$ on a field $\mathcal{F}$, we want to construct non-negative countably additive set functions $\nu$ on $\mathcal{F}$ such that $\nu|_\mathcal{F} = \mu$, i.e.

$$\nu(A) = \mu(A) \quad \forall A \in \mathcal{F}.$$

\textbf{Definition 6} (\(\sigma\)-finite measures on fields)

A measure $\mu$ on a field $\mathcal{F}$ is said to be $\sigma$-finite if $\Sigma_{n=1}^\infty A_n \in \mathcal{F}$ with $A_n \in \mathcal{F}$ and $\mu(A_n) < \infty$ for all $n$.

We now state the main result regarding the extension of measures.

\textbf{Theorem 1:} (Carathéodory Extension Theorem)

Let $\mu$ be a $\sigma$-finite measure on a field $\mathcal{F}$.

(Existence) $\mu$ has an extension to a $\sigma$-finite
measure $\tilde{\mu}$ on $\sigma(F)$, i.e.,

$$\tilde{\mu}(A) = \mu(A) \neq A \in F.$$

(Uniqueness) If $\tilde{\mu}$ and $\hat{\mu}$ are two extensions of $\mu$ as in the existence part, then they are the same, i.e.,

$$\tilde{\mu}(A) = \hat{\mu}(A) \neq A \in \sigma(F).$$

Note 3: In this course, we discuss the uniqueness part of the theorem. Existence part requires some technical construction and is not part of this course.

Uniqueness of extension

We continue with the notations and hypothesis of the theorem above. To express the argument better, we discuss the proof of the result in steps. First we discuss the result when $\mu$ is a probability measure (when $\mu(X)=1$). Then we discuss the result for the case of finite measures (when $\mu(X)<\infty$). Finally, we discuss the proof for the $\sigma$-finite case.

Proposition 3: let $\mu, \tilde{\mu}, \hat{\mu}$ and $\mathcal{F}$ be as in
Theorem 10. Further assume \( \mu \) is a probability measure on \( \mathcal{F} \), i.e., \( \mu(\mathcal{F}) = 1 \). Then

\[
\overline{\mu}(A) = \hat{\mu}(A) \neq \emptyset \in \sigma(\mathcal{F}).
\]

Proof: Consider the collection of sets

\[
\mathcal{E} := \{ A \in \sigma(\mathcal{F}) \mid \overline{\mu}(A) = \hat{\mu}(A) \}.
\]

Since \( \overline{\mu}(A) = \mu(A) = \hat{\mu}(A) \neq \emptyset \in \mathcal{F} \), we have \( \mathcal{F} \subseteq \mathcal{E} \). In particular,

\[
\overline{\mu}(\mathcal{F}) = \hat{\mu}(\mathcal{F}) = \mu(\mathcal{F}) = 1.
\]

So \( \overline{\mu} \) and \( \hat{\mu} \) are probability measures on \( \sigma(\mathcal{F}) \). By Proposition 11 of Week 2 applied to the measurable space \((\mathcal{F}, \sigma(\mathcal{F}))\), we have \( \mathcal{E} \) is a Monotone Class.

Since \( \mathcal{F} \subseteq \mathcal{E} \) with \( \mathcal{F} \) being a field on \( \mathcal{F} \), by the Monotone Class Theorem (Theorem 2 of Week 1), we have \( \sigma(\mathcal{F}) \subseteq \mathcal{E} \).

But by definition, \( \mathcal{E} \subseteq \sigma(\mathcal{F}) \). Hence \( \mathcal{E} = \sigma(\mathcal{F}) \).

Then \( \overline{\mu}(A) = \hat{\mu}(A) \neq \emptyset \in \sigma(\mathcal{F}) \).

This completes the proof.

Note 32: We now make an observation following the Monotone Class Theorem. Given
any Monotone class \( \mathcal{M} \) containing a field \( \mathcal{F} \), we have \( \sigma(\mathcal{F}) \subseteq \mathcal{M} \). But, \( \sigma(\mathcal{F}) \) is a \( \sigma \)-field and hence a Monotone class. Therefore \( \sigma(\mathcal{F}) \) is the minimal Monotone class containing the field \( \mathcal{F} \).

**Proposition 4:** let \( \mu, \bar{\mu}, \hat{\mu} \) and \( \mathcal{F} \) be as in Theorem 1. Further assume \( \mu \) is a finite measure on \( \mathcal{F} \), i.e., \( \mu(A) < \infty \). Then

\[
\bar{\mu}(A) = \hat{\mu}(A) \quad \forall A \in \sigma(\mathcal{F}).
\]

**Proof:** consider \( \bar{\mu}_1, \hat{\mu}_1 : \sigma(\mathcal{F}) \rightarrow [0, \infty] \) defined by

\[
\bar{\mu}_1(A) := \frac{\bar{\mu}(A)}{\mu(A)} ; \quad \hat{\mu}_1(A) := \frac{\hat{\mu}(A)}{\hat{\mu}(A)} \quad \forall A \in \sigma(\mathcal{F}).
\]

By Proposition 4 of Week 2 and Proposition 3 above to complete the proof. (Exercise)

**Proposition 5:** let \( \mu, \bar{\mu}, \hat{\mu} \) and \( \mathcal{F} \) be as in Theorem 1. Then

\[
\bar{\mu}(A) = \hat{\mu}(A) \quad \forall A \in \sigma(\mathcal{F}).
\]

**Proof:** Since \( \mu \) is \( \sigma \)-finite, there exists \( s_n \in \mathcal{F} \), \( \mu(s_n) < \infty \) \( \forall n \) with \( s_n \uparrow \infty \).

Since \( \bar{\mu}(A) = \hat{\mu}(A) = \bar{\mu}(A) \quad \forall A \in \mathcal{F} \), then \( \{s_n\}_{n=1}^{\infty} \)
\( \tilde{u} \) also an increasing sequence in \( \sigma(\mathcal{F}) \) with \( \omega n \uparrow \omega \), \( \tilde{\mu}(\omega n) = \tilde{\nu}(\omega n) < \infty \). Therefore, \( \tilde{\mu} \) and \( \tilde{\nu} \) are \( \sigma \)-finite measures on \( \sigma(\mathcal{F}) \).

Consider the set functions \( \tilde{\mu}_n, \tilde{\nu}_n : \sigma(\mathcal{F}) \rightarrow [0, \infty) \) defined as follows: for \( A \in \sigma(\mathcal{F}) \) and \( n = 1, 2, \ldots \)

\[ \tilde{\mu}_n(A) := \tilde{\mu}(A \cap \omega n) ; \tilde{\nu}_n(A) := \tilde{\nu}(A \cap \omega n). \]

Complete the proof by verifying the following steps (Exercise)

(i) \( \tilde{\mu}_n \) and \( \tilde{\nu}_n \) are finite measures with \( \tilde{\mu}_n(\omega) = \tilde{\nu}_n(\omega) \).

(ii) Apply Proposition 4 to show that \( \tilde{\mu}_n(A) = \tilde{\nu}_n(A) \) for all \( A \in \sigma(\mathcal{F}) \).

(iii) Use the continuity from below property for \( \tilde{\mu} \) and \( \tilde{\nu} \) to conclude the proof.

Note 32: Using the Carathéodory Extension Theorem, we shall discuss the construction of probability measures corresponding to ”distribution functions”. This discussion
will be included in the content of next week. The construction shall produce a function going in the direction opposite to Figure 2 in Note 8.