Construction of RVs with a specified law

For any real valued RV \( X \) on a probability space \((\mathbb{R}, \mathcal{F}, P)\), we have discussed the concept of its law \( P_{X^{-1}} \), which is a probability measure on \((\mathbb{R}, \mathcal{B}_\mathbb{R})\). However, as pointed out in Note 4, the association of \((\mathbb{R}, \mathcal{F}, P), X)\) to \( P_{X^{-1}} \) is not one-to-one.

In this lecture, we first discuss another function going in the opposite direction of Figure 1, i.e., given a probability measure \( \mu \) on \((\mathbb{R}, \mathcal{B}_\mathbb{R})\), we want to have a probability space \((\mathbb{R}, \mathcal{F}, P)\) and an RV \( X : (\mathbb{R}, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) such that \( P_{X^{-1}} = \mu \).

Note 25: Consider taking \( \mathcal{F} = \mathcal{B}_\mathbb{R} \), \( P = \mu \) and \( X : \mathbb{R} \to \mathbb{R} \) as \( X(x) := x \mathbb{1}_{x \in \mathbb{R}} \). Since \( X \) is a continuous function, it is Borel measurable. Check that \( P_{X^{-1}} = \mu \).
Thus, we have the following function.

\[ \{ \mu \mid \mu \text{ is a probability measure on } (\mathbb{R}, \mathcal{B}_\mathbb{R}) \} \]

**Figure 3:**

\[ P \xrightarrow{X} \mathbb{R}^d \]

\[ \mathbb{F} = \mathcal{B}_\mathbb{R} \]

\[ P = \mu \]

\[ X = \text{Identity function on } \mathbb{R} \]

\[ \{(\Omega, \mathcal{F}, P), X) \mid (\Omega, \mathcal{F}, P) \text{ is a probability space} \]

\[ X : (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \text{ is an RV} \]

**Note 26:** As mentioned in Note 4, like Figure 1, Figure 3 has its corresponding version for \( \mathbb{R}^d \)-valued random vectors. Write down the corresponding statement. (Exercise)

**Note 27:** In Figure 3, given \( \mu \), the choice of \((\Omega, \mathcal{F}, P) \) and \( X \) as considered above, is not unique.

**Note 28:** In Figures 1 and 3, we have discussed a correspondence between random variables/ vectors and probability measures on \( \mathbb{R}/\mathbb{R}^d \). We shall see that the law of an RV captures all the relevant information about the RV and as such, other than any specific
argument requiring $(\Omega, \mathcal{F}, P)$, we shall work with $(\text{IR}, \mathcal{B}_\text{IR}, P_0 \cdot \mathcal{X}) = \mu$. We may say "$X$ is an RV with law $\mu$", without mentioning $(\Omega, \mathcal{F}, P)$.

Note: We now have a way to work with all possible RVs - by simply considering all probability measures on $(\text{IR}, \mathcal{B}_\text{IR})$ or $(\text{IR}^d, \mathcal{B}_{\text{IR}^d})$. Some special examples of discrete RV are of importance.

(i) A constant or degenerate RV $X$ has the law $\delta_c$ for some $c \in \text{IR}$.

(ii) A Bernoulli RV $X$ has the law

$p \delta_1 + (1-p) \delta_0$ for some $p \in (0,1)$. Here,

$P(X=0) = P_0(X^I(0)) = (p \delta_1 + (1-p) \delta_0)(0)$

$= 1-p$

and

$P(X=1) = P_0(X^I(1)) = (p \delta_1 + (1-p) \delta_0)(1)$

$= p$.

(iii) A Binomial $(n,p)$ RV $X$ has the law

$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta_k$. For $i=0,1,\ldots,n$,

$P(X=i) = P_0(X^I(i)) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta_k(i)$
\[(i)\text{ A Poisson}(\lambda) \text{ RV } X \text{ has the law}\]

\[
\sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \delta_k. \text{ For } i = 0, 1, \ldots
\]

\[
\Pr(X = i) = \text{Po}(\lambda)(\{i\}) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \delta_k (\{i\}) = e^{-\lambda} \frac{\lambda^i}{i!}.
\]

Similarly, the laws of standard discrete RVs may be written as some convex linear combination of the Dirac measures.

**Note 30**: In a later lecture, we shall discuss about the laws of continuous RVs.

**Exercise 3**: Compute the distribution functions of the RVs mentioned in Note 29.