Random Variables and Random Vectors

In the previous lectures, we discussed the concept of Borel measurable functions on measurable spaces $(\Omega, \mathcal{F})$.

We now consider Borel measurable functions in the following special case.

Definition 5 (Random Variables and Random Vectors)

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Any Borel measurable function $X: (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R})$ is said to be a real-valued random variable or univariate random variable or simply, a random variable.

Any Borel measurable function $X: (\mathbb{R}, \mathcal{F}) \rightarrow (\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ is said to be an $\mathbb{R}^d$-valued random vector or $\mathbb{R}^d$-valued random variable.

Note 24: If the dimension $d$ is clear from the context, we may use the term random vector to mean an $\mathbb{R}^d$-valued
random vector. Of course, an \( \mathbb{R} \)-valued random vector is just a random variable.

**Note 25:** By Proposition 3, we have:

\[
x = (x_1, x_2, ..., x_d)^t
\]

is a random vector if and only if \( x_i, i=1,2, ..., d \) are random variables.

**Note 26:** To simplify the terminology further, we may use 'RV' to mean a random variable.

**Note 27:** So far, in defining RVs, the probability measure on the domain side is not playing any explicit role. We shall see its usage soon. In fact, this shall allow us to construct all possible RVs.

**Note 28:** In relation to Note 27, we do not immediately discuss explicit examples of RVs. However, as long as we can define/equip a probability measure on the domain, any Borel measurable function becomes an RV. In order to highlight the probability measure on the
domain, we shall use statements like 
\[ X : (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \text{ is an RV}. \]

**Note:** To make a distinction between Borel measurable functions and random variables, we use small letters \( f, g \) etc. to denote the Borel measurable functions, and capital letters \( X, Y \) etc. to denote random variables/ vectors.

Using Theorem 1, we have the following result.

**Theorem 6:** Let \( X \) and \( Y \) be RVs defined on the space \((\Omega, \mathcal{F}, P)\). Then \( X + Y, X - Y, X \cdot Y, |X|, X \wedge Y, X \vee Y, X^+, X^- \) are also RVs defined on the same probability space.

Using Theorem 2, we have the following result.

**Theorem 6:** Let \( X \) and \( Y \) be RVs defined on the space \((\Omega, \mathcal{F}, P)\). Further assume that \( Y \) takes values in \( \mathbb{R} \setminus \{0\} \). Then \( \frac{X}{Y} \) is also an RV defined on the same probability space.
Combining Theorem 5 and Note 25, we obtain the next result.

**Theorem 7**: Let \( X = (X_1, \ldots, X_d)^t \) and \( Y = (Y_1, \ldots, Y_d)^t \) be random vectors defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Then so are \( X+Y = (X_1+Y_1, \ldots, X_d+Y_d)^t \) and \( X-Y = (X_1-Y_1, \ldots, X_d-Y_d)^t \).

**Note 30**: In fact, applying Theorem 5 for the component RVs in Theorem 7, we can list many more examples of random vectors.

Following the ideas discussed in Proposition 2 and Theorem 1, we have the next result.

**Theorem 8**: Let \( X = (X_1, \ldots, X_d)^t \) be an \( \mathbb{R}^d \)-valued random vector on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( f = (f_1, \ldots, f_p) : (\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d}) \to (\mathbb{R}^p, \mathcal{B}_{\mathbb{R}^p}) \) be Borel measurable with component functions \( f_1, \ldots, f_p \). Then \( Y = f \circ X \) is an \( \mathbb{R}^p \)-valued random vector with component RVs \( Y_i = f_i \circ X, \ i=1,2,\ldots, p \).
Recall the generating classes of $\mathcal{B}_\mathbb{R}$ from Week 1. Using Note 13, we have the following criteria to check if a function is an RV.

**Theorem 5:** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X: \Omega \rightarrow \mathbb{R}$ be a function. Then $X$ is an RV if and only if $X^{-1}((-\infty, x]) \in \mathcal{F}$ for all $x \in \mathbb{R}$. Another equivalent condition is $X^{-1}((a, b]) \in \mathcal{F}$ for $-\infty \leq a < b \leq \infty$.

**Note 31:** Recall that $X^{-1}(A) = \{ \omega \in \Omega | X(\omega) \in A \}$, $\forall A \in \mathcal{B}_\mathbb{R}$ for any RV defined on $(\Omega, \mathcal{F}, P)$. It is convenient to suppress “$\omega$” in the statement above and write $(X \in A)$ to denote the same event. In all subsequent discussions, we shall use the following expressions interchangeably:

$$X^{-1}(A) = \{ \omega \in \Omega | X(\omega) \in A \} = (X \in A).$$

**Note 32:** Given an RV $X: (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R})$, by Exercise 1 of Week 1, we have
$\bar{X}'(\mathcal{B}_R) = \{ \bar{X}'(A) \mid A \in \mathcal{B}_R \}$ is a $\sigma$-field. This $\sigma$-field captures all the events involving the RV $X$ and in future discussions, we denote this $\sigma$-field by $\sigma(X)$. Since $\bar{X}'(A) \in \mathcal{F}$ for all $A \in \mathcal{B}_R$, by definition $\sigma(X) \subseteq \mathcal{F}$, i.e. $\sigma(X)$ is a sub-$\sigma$-field of $\mathcal{F}$.

Exercise 8: Let $X: (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}_R)$ be an RV.

(i) Show that $(\Omega, \sigma(X), P)$ is a probability space.

(ii) If $Y: (\Omega, \sigma(X), P) \to (\mathbb{R}, \mathcal{B}_R)$ be an RV, then show that there exists $f: (\mathbb{R}, \mathcal{B}_R) \to (\mathbb{R}, \mathcal{B}_R)$ measurable, such that $Y = f \circ X = f(X)$. (Hint: First take $Y$ to be an indicator function. Can you find $f$ in this case?)