Limiting behaviour of measurable functions

In the previous lecture, we discussed about algebraic properties of measurable functions on a measurable space \((\mathbb{R}, \mathcal{F})\). We now consider sequences of functions \(\{f_n\}_n\) on \((\mathbb{R}, \mathcal{F})\) and study their limiting behaviour.

Suppose that \(f_n : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})\) is measurable for each \(n\). Now, consider functions of the form \(g, h : \mathbb{R} \to \mathbb{R}\) given by

\[
g(x) := \lim_{n \to \infty} f_n(x), \quad x \in \mathbb{R}
\]

(if the limit exists)

\[
h(x) := \sup_n f_n(x), \quad x \in \mathbb{R}.
\]

Even though the functions \(f_n\) are \(\mathbb{R}\)-valued, the functions \(g\) and \(h\) may take values in \(\overline{\mathbb{R}}\). As such we may as well take \(f_n\)'s to be \(\overline{\mathbb{R}}\)-valued and then discuss properties of the functions \(g, h\) etc.

**Theorem:** Let \(f_n : (\mathbb{R}, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})\) be Borel measurable for all \(n\) and assume that
\[ \lim_{n \to \infty} f_n(w) \] exists in \( \overline{R} \) for all \( w \in R \). Then the function \( g : (R, \mathcal{F}) \to (\overline{R}, \mathcal{B}(\overline{R})) \) defined by \[ g(w) := \lim_{n \to \infty} f_n(w), \quad w \in R \] is Borel measurable.

**Proof:** Since the sets \( (x, \infty] \), \( x \in R \) generate \( \mathcal{B}(\overline{R}) \), using Note 13, we need to check if
\[ g^{-1}((x, \infty]) = \{ w \in R | g(w) > x \} \in \mathcal{F} \] \( \forall x \in R \).

Now,
\[ \{ w \in R | g(w) > x \} = \bigcap_{n=1}^{\infty} \{ w \in R | \lim_{n \to \infty} f_n(w) > x \} \]
\[ = \bigcup_{m=1}^{\infty} \bigcap_{n \to \infty} \{ w \in R | f_n(w) > x + \frac{1}{m} \} \]
\[ = \bigcup_{m=1}^{\infty} \liminf_{n \to \infty} \{ w \in R | f_n(w) > x + \frac{1}{m} \} \]
\[ = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{ w \in R | f_k(w) > x + \frac{1}{m} \} \]

Since \( f_k \)'s are one Borel measurable, the sets \( \{ w \in R | f_k(w) > x + \frac{1}{m} \} = g_k^{-1}((x + \frac{1}{m}, \infty]) \in \mathcal{F} \).

Hence, \( g^{-1}((x, \infty]) \in \mathcal{F} \) \( \forall x \in R \). This completes the proof.

**Note 15:** In the above theorem, if \( f_n \)'s are one \( IR \) valued Borel measurable functions and if
\[ \lim_{n \to \infty} f_n(w) \text{ exists in } \mathbb{R} \text{ for all } w \in \mathbb{R}, \text{ then the same argument shows } g'(x,\infty) \in \mathcal{F}, \text{ for all } x \in \mathbb{R}. \text{ In this case } g : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R}) \text{ becomes Borel measurable.} \]

Proof of the next result is similar to Theorem 2. We skip the details for brevity.

**Theorem 4:** Let \( f_n : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) be Borel measurable for all \( n \). Show that the functions \( \lim_{n} \sup f_n \), \( \lim_{n} \inf f_n \), \( \sup_n f_n \) and \( \inf_n f_n \) are also \( \mathbb{R} \)-valued Borel measurable functions.

**Exercise 6:** Let \( g : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) be Borel measurable. If \( \text{Range}(g) \subseteq \mathbb{R} \), then show that \( g : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) is measurable. (Hint: Compare with Exercise 5)

**Note 6:** Let \( f_n : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) be Borel measurable for all \( n \). If the functions \( \lim_{n} \sup f_n \), \( \lim_{n} \inf f_n \), \( \sup_n f_n \) and \( \inf_n f_n \) are \( \mathbb{R} \)-valued, then they are also \( \mathcal{F}/\mathcal{B}_\mathbb{R} \) measurable.
Note 17: Recall the functions $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ discussed in Theorem 1.

These functions are referred to as the "positive part of $f$" and "negative part of $f$" respectively. The following statements are left as exercise.

(a) $f^+(\omega) \geq 0$ and $f^-(\omega) \geq 0 \quad \forall \omega \in \Omega$.

(b) For some $\omega$, $f^+(\omega) > 0$
implies $f^-(\omega) = 0$ for that $\omega$.

(c) For some $\omega$, $f^-(\omega) > 0$
implies $f^+(\omega) = 0$ for that $\omega$.

(d) $\{\omega \in \Omega | f^+(\omega) > 0\} \cap \{\omega \in \Omega | f^-(\omega) > 0\} = \emptyset$.

(e) $f(\omega) = f^+(\omega) - f^-(\omega) \quad \forall \omega \in \Omega$. Since Statement (d) holds, "$\infty - \infty"$ situation does not appear on the right hand side of the equality.

(f) $|f(\omega)| = f^+(\omega) + f^-(\omega) \quad \forall \omega \in \Omega$.

Definition 4 (Simple functions)

let $(\Omega, \mathcal{F})$ be a measurable space. A Borel measurable function $f: \Omega \to \mathbb{R}$ (or $f: \Omega \rightarrow \mathbb{R}$) is said to be a Simple function
if it can be written as \( \sum_{i=1}^{n} x_i 1_{A_i} \) where
\( A_1, A_2, \ldots, A_n \in \mathcal{F} \) and \( x_1, x_2, \ldots, x_n \in \mathbb{R} \) (or \( \mathbb{R} \))
with \( A_i \)'s being pairwise disjoint.

Note 18: If \( A, B \in \mathcal{F} \), with \( A \cap B \neq \emptyset \), then
observe that \( 1_A + 1_B = 1_{A \setminus B} + 2 \cdot 1_{A \cap B} + 1 \), with the sets appearing on the
right hand side being pairwise disjoint.

More generally, given a finite linear
combination of indicator functions, we
can rewrite the function as a linear
combination of indicator functions with
the sets being pairwise disjoint. As a
consequence, as long as the addition of
two simple functions is defined, (i.e. \( \infty - \infty 
\) does not appear) we obtain a simple
function.

Note 19: Recall that \( 1_A : \mathbb{R} \to \mathbb{R} \) is
Borel measurable if and only if \( A \in \mathcal{F} 
\). Given \( c \in \overline{\mathbb{R}} \), the constant function \( c 
\) is also Borel measurable. Thus, by
Theorem 1, we have the measurability of the product $c_{1A}$, provided $A \in \mathcal{F}$.

Again by Theorem 1, $\sum_{i=1}^{n} x_i 1_{A_i}$, with $A_i$'s in $\mathcal{F}$ and $x_i$'s in $\mathbb{R}$ (or $\mathbb{R}$), is Borel measurable.

Note 20: Without loss of generality, we may assume $\bigcup_{i=1}^{n} A_i = \mathbb{R}$. If $\bigcup_{i=1}^{n} A_i \subseteq \mathbb{R}$, then taking $A_{n+1} = \left( \bigcup_{i=1}^{n} A_i \right)^c$, we can rewrite the simple function as $\sum_{i=1}^{n} x_i 1_{A_i} + 0 \cdot 1_{A_{n+1}}$. By definition, a simple function only takes the finitely many values $x_1, x_2, \ldots, x_n$.

Note 21: The values $\pm \infty$ are allowed for simple functions, provided $\sum_{i=1}^{n} x_i 1_{A_i}$ makes sense. However, in practice, we work with simple functions which take values in $\mathbb{R}$.

Theorem 4 (Approximation by Simple Functions)

Let $f : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ be Borel measurable.

(i) If $f$ takes only non-negative values, then there exists a sequence $\{f_n\}_n$ of
$\mathbb{R}$-valued, non-negative, simple functions such that $f_n(\omega) \leq f_{n+1}(\omega)$ for all $n, \omega$ and
\[ \lim_{n \to \infty} f_n(\omega) = f(\omega) \quad \forall \omega. \]
In this case, we write $f_n \uparrow f$.

(iii) In general, the function $f$ is the limit of a sequence $\{f_n\}$ of $\mathbb{R}$-valued, simple functions such that $|f_n(\omega)| \leq |f(\omega)|$ for all $n, \omega$.

Proof: (i) Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by

\[ f_n(\omega) := \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(\omega) < \frac{k}{2^n} \\ n & \text{if } f(\omega) \geq n. \end{cases} \]

Then,

\[ f_n(\omega) \]

\[ = \begin{cases} \frac{k-1}{2^n} & \text{if } \omega \in f_1\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]\right) \\ n & \text{if } \omega \in f_1\left([n, \infty]\right). \end{cases} \]

\[ = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]}(\omega) + n 1_{f_1\left([n, \infty]\right)}(\omega). \]

The verification of the relevant properties is straightforward.
(ii) Recall from Note 17 that \( f = f^+ - f^- \) with \( f^+ \geq 0 \) and \( f^- \geq 0 \). For \( f^+ \) and \( f^- \), using part (i), construct sequences \( \{g_n\}_n \) and \( \{h_n\}_n \) with \( g_n \uparrow f^+ \) and \( h_n \uparrow f^- \). Since,

\[ 0 \leq g_n \leq f^+ \quad \text{and} \quad 0 \leq h_n \leq f^- \]

by statement (d) of Note 17 above, we have

\[ \{ \omega \mid g_n(\omega) > 0 \} \cap \{ \omega \mid h_n(\omega) > 0 \} = \emptyset + n \]

and hence \( f(\omega) = \lim_{n \to \infty} [g_n(\omega) - h_n(\omega)] + \omega \).

Here, “\(-\infty - \infty\)” situation does not appear on the right hand side.

Take \( f_n := g_n - h_n + n \). Check that \( f_n \) is simple. Finally,

\[ |f_n| = g_n + h_n \quad \text{(why?)} \]

\[ \leq f^+ + f^- = |f| \]

This completes the proof.

Note 22: If \( f \) is bounded, then in Theorem 4, the pointwise convergence of \( \{f_n\}_n \) to \( f \) is uniform.

Exercise 7: Let \( \{A_n\}_n \) be an increasing sequence of sets in a \( \sigma \)-field \( \mathcal{F} \) and let \( A = \bigcup_{n=1}^{\infty} A_n \). Is it true that
\[ \lim_{n \to \infty} \mathbb{1}_A \]

Note (23): While proving some general properties of measurable functions, the following approach is quite useful. Prove the relevant property in the following order: first for indicator/simple functions, then for non-negative measurable functions, and finally for measurable functions.