In the last two lectures, we have discussed the concept of Borel measurable functions. In this lecture, we focus on algebraic properties of such functions. We shall also see that these properties allow us to construct more examples from the existing examples.

**Proposition 2:** Let \( f : (\mathbb{R}, F_1) \to (\mathbb{R}, F_2) \) and \( g : (\mathbb{R}, F_2) \to (\mathbb{R}, F_3) \) be measurable functions. Then the composition \( g \circ f : (\mathbb{R}, F_1) \to (\mathbb{R}, F_3) \) is also measurable.

**Proof:** Let \( A \in F_3 \). We need to check

\[
(g \circ f)^{-1}(A) \in F_1.
\]

By the measurability of \( g \), \( g^{-1}(A) \in F_2 \).

By the measurability of \( f \),

\[
(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \in F_1.
\]

This completes the proof.

**Proposition 3:** Let \( (\mathbb{R}, F) \) be a measurable
Space. Let \( f: \mathbb{R} \rightarrow \mathbb{R}^d \) be a function and write the component functions as 
\( f = (f_1, f_2, \ldots, f_d) \). Then \( f \) is Borel measurable if and only if \( f_i, i=1,2,\ldots,d \) are also Borel measurable.

**Proof**: Recall from Note 13 of the previous lecture that \( \Pi_i: (\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d}) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \) \( i=1,2,\ldots,d \) are Borel measurable.

If \( f \) is Borel measurable, then by Proposition 2 above, \( f_i = \Pi_i \circ f, i=1,2,\ldots,d \) are also Borel measurable.

To prove the converse statement. First recall the generating class of \( \mathcal{B}_{\mathbb{R}^d} \) mentioned in Notes of Week 1. Again using Note 13 of the previous lecture, we only verify the measurability condition for the generating sets on the range \( \mathbb{R}^d \).

We have, for \(-\infty \leq a_i < b_i \leq \infty, i=1,2,\ldots,d\)

\[
\overline{f}^{-1}\left( \prod_{i=1}^{d} (a_i, b_i] \right) = \bigcap_{i=1}^{d} \overline{f_i}^{-1}(a_i, b_i].
\]
But the right hand side is in $\mathcal{F}$, since $\mathcal{F}$'s are Borel measurable. This completes the proof.

Exercise 4: Prove the analogous version of Proposition 3 by replacing all $\mathbb{R}$ with $\mathbb{R}^*$.

We now discuss the main result of this lecture.

**Theorem 1:** Let $f, g : (\mathbb{N}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ be Borel measurable. Then so are $f + g$, $f - g$, $fg$, $|f|$, $f \land g = \min \{f, g\}$, $f \lor g = \max \{f, g\}$, $f^+ = \max \{f, 0\}$, $f^- = \max \{-f, 0\}$.

**Proof:** We prove the result for $f + g$. The argument for rest of the functions is similar and is left as an exercise.

Since $f, g : (\mathbb{N}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ are Borel measurable, by Proposition 3, $(f, g) : (\mathbb{N}, \mathcal{F}) \to (\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$ is Borel measurable.

Note that the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by $h(x, y) := x + y \uparrow (x, y) \in \mathbb{R}^2$ is
a continuous function. Hence by Note 13

\[ h: (\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2}) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \] is measurable.

Then by Proposition 2 above,

\[ f + g = h \circ (f, g): (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \] is Borel measurable. This completes the proof.

Note 14: Recall from Week 1 about the Borel \(0\)-fields on Borel subsets of \(\mathbb{R}\). In the next result we require the Borel \(0\)-field on \(\mathbb{R}\backslash \{0\}\).

Theorem 2: Consider \(f\) as in Theorem 1 and take a measurable \(g: (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \rightarrow (\mathbb{R}\backslash \{0\}, \mathcal{B}_{\mathbb{R}\backslash \{0\}})\).

Then \(f/g: \mathbb{R}\backslash \{0\} \rightarrow \mathbb{R}\) is defined and is \(f/g\) measurable.

Proof: Since \(g\) takes non-zero values, the function \(f/g\) is well-defined.

Observe that the function

\[ \mathbb{R} \times (\mathbb{R}\backslash \{0\}) \rightarrow \mathbb{R} \\
(x, y) \mapsto \frac{x}{y} \]

is continuous. The proof now follows.
Similar to the argument in Theorem 1, complete the proof. (Exercise)

Exercise 5: Consider the following variant of the hypothesis of Theorem 2. Let $g : (\mathbb{R}, \mathcal{F}) \rightarrow (\mathbb{R}/\mathbb{Q}, \mathcal{B})$ be measurable with Range $(g) \subseteq \mathbb{R}\setminus\{0\}$. Is $g : (\mathbb{R}, \mathcal{F}) \rightarrow (\mathbb{R}\setminus\{0\}, \mathcal{B})$ measurable?