Borel measurable functions

In the previous lecture, we discussed the concept of $\mathbb{R}$-valued measurable functions, i.e., the Borel measurable functions. However, in practice, we sometimes need to work with functions taking values in $\mathbb{R}^d$ or $\overline{\mathbb{R}}$. We now define the concept $\mathbb{R}^d$ or $\overline{\mathbb{R}}$ valued measurable functions.

Note 7: Recall that the Borel $\sigma$-fields for $\mathbb{R}^d$ and $\overline{\mathbb{R}}$ were discussed in Week 1.

Note 8: We may also consider functions taking values in $\overline{\mathbb{R}}^d$. In this course, we do not discuss such functions.

Definition 3 (\(\mathbb{R}^d\) or \(\overline{\mathbb{R}}\) valued Borel measurable functions)

Take $\mathcal{S} = \mathbb{R}^d$ or $\overline{\mathbb{R}}$. Let $(\mathcal{S}, \mathcal{F})$ be a measurable space. A function $f :\mathcal{S} \rightarrow \mathcal{S}$ is said to be Borel measurable if it is $\mathcal{F} \otimes \mathcal{F}$ measurable.
Note 9: For notational convenience, we use the term \"Borel measurable\" to refer to \(\mathbb{R}, \mathbb{R}^d\) or \(\overline{\mathbb{R}}\) valued functions measurable with respect to the appropriate Borel \(\sigma\)-fields. Here, we are combining Definition 2 and Definition 3.

Note 10: Continuing with the discussion in Note 5, we use the notation/statement \"\(f : (\mathbb{R}, \mathcal{B}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})\) is measurable\" or \"\(f : \mathbb{R} \to \overline{\mathbb{R}}\) is Borel measurable\" to mean the Borel measurability of \(f\) when \(\overline{\mathbb{R}}\) is \(\mathbb{R}, \mathbb{R}^d\) or \(\overline{\mathbb{R}}\). Since we are interested mostly in Borel measurable functions, we may simply say measurable functions, instead of saying Borel measurable functions. However, in any such discussion, the \(\sigma\)-fields should be clearly stated.

Note 11: As discussed in the previous lecture, the constant functions remain
Borel measurable, when the range is \( \mathbb{R}^d \) or \( \overline{\mathbb{R}} \).

**Note 1:** We are interested in the non-constant Borel measurable functions taking values in \( \mathbb{R}^d \) or \( \overline{\mathbb{R}} \).

**Exercise 3:** Show that all \( \mathbb{R} \)-valued Borel measurable functions are \( \overline{\mathbb{R}} \)-valued Borel measurable.

**Note 2 (i):** Using the Principle of Good Sets (see Note 2 of Week 1), we can prove the following statement:

Let \( (\mathcal{X}_1, \mathcal{F}_1) \) and \( (\mathcal{X}_2, \mathcal{F}_2) \) be measurable spaces. Let \( \mathcal{F}_2 = \mathcal{G}(\mathcal{C}) \) for some collection \( \mathcal{C} \) of \( \mathcal{X}_2 \) subsets of \( \mathcal{X}_2 \). Then \( f: \mathcal{X}_1 \rightarrow \mathcal{X}_2 \) is \( \mathcal{F}_1 \)-\( \mathcal{F}_2 \) measurable if and only if

\[
\tilde{f}^{-1}(A) \in \mathcal{F}_1 \quad \forall A \in \mathcal{C}.
\]

The statement suggests that looking at the pre-images of the generating sets on the range side is enough to ascertain the measurability of \( f \).
(ii) This observation can be used to prove Proposition 1 discussed in the previous lecture.

(iii) We also have the following extension of Proposition 1 to higher dimensions:

let \( f: \mathbb{R}^n \to \mathbb{R}^m \) be a continuous function. Then \( f: (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}) \to (\mathbb{R}^m, \mathcal{B}_{\mathbb{R}^m}) \) is Borel measurable.

(iv) Part (iii) above yields a large class of examples of Borel measurable functions. In particular, consider the co-ordinate projection maps \( \pi_i: \mathbb{R}^d \to \mathbb{R}, \ i=1,2,\ldots,d \) defined by \( \pi_i((x_1, x_2, \ldots, x_d)^t) = x_i \) for all \((x_1, \ldots, x_d)^t \in \mathbb{R}^d\).

Since \( \pi_i \)'s are continuous, they are Borel measurable. We are going to use these projection maps in the next lecture.