**Qn 1.**

Suppose the sphere $S^2$ is covered by a single surface patch $\sigma: U \rightarrow S^2$, $U \subseteq \mathbb{R}^2$ open.

Then $S^2$ is homeomorphic to $U$. — (1)

Thus $U$ is compact hence closed in $\mathbb{R}^2$. — (1)

In $\mathbb{R}^2$ a set which is both open and closed in either $\emptyset$ or $\mathbb{R}^2$. But $\mathbb{R}^2$ can not be compact. — (3)

**Qn 2.**

(i) **Plane**

$\sigma(u, \varphi) = a + u \overrightarrow{b} + \varphi \overrightarrow{v}$

$\overrightarrow{b}$ and $\overrightarrow{v}$ for perpendicular unit vectors

$\sigma_u \times \sigma_\varphi = \overrightarrow{b} \times \overrightarrow{v} \neq 0$ — (2)

(ii) **Sphere**

$\sigma(\vartheta, \phi) = (\cos \vartheta \cos \phi, \cos \vartheta \sin \phi, \sin \vartheta)$

$\sigma_\vartheta = (-\sin \theta \cos \phi, -\sin \theta \sin \phi, \cos \theta)$

$\sigma_\phi = (-\cos \theta \sin \phi, \cos \theta \cos \phi, 0)$

$\| \sigma_\vartheta \times \sigma_\phi \| = |\cos \theta|$, $\theta \in (-\pi/2, \pi/2)$

$\neq 0$ — (2)

(iii) **Level Surface of a Smooth Func $f: \mathbb{R}^n \rightarrow \mathbb{R}$**

$f(x, y) = (x, y, f(x, y))$

$\sigma_x \times \sigma_y = (-f_x, f_y, 1) \neq 0$ — (2)
(iv) Ellipsoid \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \] is a level surface of \( f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \)

\( (f_x, f_y, f_z) = \left( \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right) \)

vanishes only at \((0, 0, 0)\) because nowhere on the ellipsoid. Use Theorem done in Lecture.

(2) \[ \text{Alt.: Ellipsoid can be parametrized as} \]

\[ \Gamma(\theta, \phi) = (a \cos \theta \cos \phi, b \cos \theta \sin \phi, c \sin \theta) \]

\( \theta \in (-\pi, \pi), \phi \in (0, \pi) \)

(one needs two surface patches because)

\[ \text{Show} \quad \| \Gamma_0 \times \Gamma_\phi \| \neq 0 \quad - (2) \]

(v) Torus can be parametrized as

\[ \Gamma(\theta, \phi) = \left( (a + b \cos \theta) \cos \phi, (a + b \cos \theta) \sin \phi, b \sin \theta \right) \]

with four patches:

(i) \( 0 < \theta < 2\pi, \quad 0 < \phi < 2\pi \)

(ii) \( 0 < \theta < 2\pi, \quad -\pi < \phi < \pi \)

(iii) \( -\pi < \theta < \pi, \quad 0 < \phi < 2\pi \)

(iv) \( -\pi < \theta < \pi, \quad -\pi < \phi < \pi \)

\[ \Gamma_0 \times \Gamma_\phi = -b(a + b \cos \theta) \left( \cos \phi \cos \theta, \cos \phi \sin \theta, 1 \right) \]

(2) \[ \text{Alt.: Torus is level surface of} \]

\[ (x^2 + y^2 + z^2 - a^2)^2 = 4a^2(x^2 + y^2) \]

Find \( (f_x, f_y, f_z) \) - (2)
3. Parametrization is straightforward to check.

\[ U = \left\{ (r, \theta) \in \mathbb{R}^2 : r > 0 \right\} \quad - (5) \]

4. At \((1, 0, 1)\), \(r = 1\) and \(\theta = 0\)

\[ \vec{r} = (1, 0, 2) \quad \vec{\theta} = (0, 1, 2) \quad - (2) \]

\[ \vec{r} \times \vec{\theta} = (-2, -2, 1). \]

Equation of tangent plane: \(-2x - 2y + z = 0\) - (3)

5. Let \(\vec{\sigma}(\vec{u}, \vec{v})\) be a reparametrization of \(\sigma(u, v)\).

\[ \vec{\sigma}_u = \frac{\partial \vec{\sigma}}{\partial u} \quad \vec{\sigma}_v = \frac{\partial \vec{\sigma}}{\partial v} \quad - (2) \]

\[ \mathcal{C} = \text{span} \left\{ \vec{\sigma}_u, \vec{\sigma}_v \right\} = \text{span} \left\{ \vec{\sigma}_u \times \vec{\sigma}_v \right\}. \quad - (3) \]