Curves and Surfaces : Solutions to Assignment 2

Qn 1.

Suppose the sphere S^2 is covered by a single surface patch $\sigma : U \to S^2$, $U \subseteq \mathbb{R}^2$ open

Then S^2 is homeomorphic to U. — (1)

Thus U is compact hence closed in $\mathbb{R}^2$. — (1)

In $\mathbb{R}^2$ a set which is both open and closed in either ϕ or $\mathbb{R}^2$. But $\mathbb{R}^2$ can not be compact — (3)

Qn 2 (i) plane $\sigma(u, v) = \vec{a} + u\vec{p} + v\vec{q}$

$\vec{p}$ and $\vec{q}$ perpendicular unit vectors

$\sigma_u \times \sigma_v = \vec{p} \times \vec{q} \neq 0$ — (2)

(ii) sphere $\sigma(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$

$\sigma_\theta = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta)$

$\sigma_\phi = (-\cos\phi\sin\phi \quad \cos\theta\cos\phi, 0)$

$\|\sigma_\theta \times \sigma_\phi\| = |\cos\theta|$, $\theta \in (-\pi/2, \pi/2)$

$\neq 0$ (2)

(iii) Level surface of a smooth fn. $f : \mathbb{R}^2 \to \mathbb{R}$

$\sigma(x, y) = (x, y, f(x, y))$

$\sigma_x \times \sigma_y = (-f_x, f_y, 1) \neq 0$ — (2)

(iv) ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. a, b, c non-zero constants

is level surface $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

$$(f_x, f_y, f_z) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$$

vanishes only at $(0, 0, 0)$ hence nowhere on the ellipsoid. Use Theorem done in Lecture. (2)

<u>Alt.</u> ellipsoid can be parametrized as

$$\sigma(\theta, \phi) = (a\cos\theta\cos\phi,\ b\cos\theta\sin\phi,\ c\sin\theta)$$

$\theta \in (-\pi/2, \pi/2)$ $\quad \phi \in (0, 2\pi)$ -

(one needs two surface patches; ~~find the~~ ~~another~~ another).

show $\|\sigma_\theta \times \sigma_\phi\| \neq 0$ — (2)

(v) Torus can be parametrized as

$$\sigma(\theta, \phi) = ((a + b\cos\theta)\cos\phi,\ (a + b\cos\theta)\sin\phi,\ b\sin\theta)$$

with four patches

(i) $0 < \theta < 2\pi, \quad 0 < \phi < 2\pi$

(ii) $0 < \theta < 2\pi, \quad -\pi < \phi < \pi$

(iii) $-\pi < \theta < \pi, \quad 0 < \phi < 2\pi$

(iv) $-\pi < \theta < \pi, \quad -\pi < \phi < \pi$

$$\sigma_\theta \times \sigma_\phi = -b(a + b\cos\theta)(\cos\theta\cos\phi, \cos\theta\sin\phi, 1)$$

$$\neq 0$$ — (2)

<u>Alt.</u> Torus is level surface of

$$(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$$ — (2)

Find (f_x, f_y, f_z)

Qn 3. Parametrization is ~~straightforward~~ to check.

$U = \{ (r, \theta) \in \mathbb{R}^2 : r > 0 \}$ — (5)

Qn 4 At $(1,0,1)$, $r = 1$ and $\theta = 0$

$\Rightarrow \quad \sigma_r = (1,0,2) \quad \sigma_\theta = (0,1,2)$ — (2)

$\sigma_r \times \sigma_\theta = (-2, -2, 1)$.

eqn. of tangent plane. $-2x - 2y + z = 0$ — (3)

Qn 5. Let $\tilde{\sigma}(\tilde{u}, \tilde{v})$ be a reparametrization of $\sigma(u, v)$.

$\sigma_u = \frac{\partial \tilde{u}}{\partial u} \tilde{\sigma}_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial u} \tilde{\sigma}_{\tilde{v}}$ —

$\sigma_v = \frac{\partial \tilde{u}}{\partial v} \tilde{\sigma}_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial v} \tilde{\sigma}_{\tilde{v}}$. — (2)

$\Rightarrow \text{span}\{\sigma_u, \sigma_v\} = \text{span}\{\tilde{\sigma}_{\tilde{u}}, \tilde{\sigma}_{\tilde{v}}\}$. — (3).