Solutions to Assignment-4

Qn 1. Let N be the standard unit normal to σ

$L = M = N = 0$ Recall $\left.\begin{aligned} L &= \sigma_{uu} \cdot N \\ M &= \sigma_{uv} \cdot N \\ N &= \sigma_{vv} \cdot N \end{aligned}\right\} \quad (*)$

Now $\sigma_u \cdot N = 0$

$\Rightarrow \sigma_u \cdot N_u + \sigma_{uu} \cdot N = 0$

$\Rightarrow \sigma_u \cdot N_u = 0$ – from $(*)$. — (2)

Similarly, $N_u \cdot \sigma_v = 0 = N_v \cdot \sigma_u = N_v \cdot \sigma_v$

Thus N_u and $N_v \perp \operatorname{span}\{\sigma_u, \sigma_v\}$

$\Rightarrow N_u$ and N_v are parallel to N

But $\|N\|^2 = 1 \Rightarrow 2 \cdot N_u \cdot N = 0 = 2 N_v \cdot N$

$\Rightarrow N_u$ is $\perp N$, $N_v \perp N$

$\Rightarrow N_u = N_v = 0$

$\Rightarrow N$ is constant. — (2)

$\Rightarrow (\sigma \cdot N)_u = \sigma_u \cdot N + \sigma \cdot N_u = 0$

$(\sigma \cdot N)_v = 0$

$\Rightarrow \sigma \cdot N =$ constant

$\Rightarrow \sigma$ is part of a plane $\vec{r} \cdot N = c$ — (2)

Qn 2

unit normal $N = (-\dot{g}\cos\vartheta,\ -\dot{g}\sin\vartheta,\ \dot{f})$

(dot wrt $\frac{d}{du}$). — (1)

a) $\vartheta = \vartheta_0$ constant

$\sigma_u = (\dot{f}\cos\vartheta_0,\ \dot{f}\sin\vartheta_0,\ \dot{g})$ - unit vector — (1)

$k_g = \sigma_{uu} \cdot (N \times \sigma_u) = 0$ — (1)

b) $u = u_0$ constant

$\sigma_v = (-f\sin v,\ f\cos v,\ 0)$ - not unit vector — (1)

arc length $s = f(u)\vartheta$ — (1)

$k_g = \frac{1}{f(u)^2}\sigma_{vv} \cdot \left(N \times \frac{1}{f(u)}\sigma_v\right) = \frac{\dot{f}}{f}$ — (1)

Qn 3 Suppose $k_1 \neq k_2$. Wlog we take $k_1 > k_2$

By Euler's Thm $K_n = k_1\cos^2\theta + k_2\sin^2\theta$

$= k_1 - (k_1 - k_2)\sin^2\theta$

$\Rightarrow$ $K_n \le k_1$ with equality iff

$\theta = 0$ or $\pi \rightarrow (*)$ — (2)

Similarly $K_n \ge k_2$ — $(**)$ (2).

If $k_1 = k_2 = k$ $k_n = k$ by Euler's Thm. → (2)

$(*)$ that is, iff $\dot{\gamma}$ is parallel to Principal vector $\vec{t_1}$

$(**)$ that is $\dot{x}$ is parallel to $\vec{t_2}$.

Qn 4.

Gaussian curvature $K_G = -\frac{\ddot{f}}{f}$

a) $K_G \equiv 0$; $\Rightarrow \ddot{f} = 0 \Rightarrow f(u) = au + b$

Since $\dot{f}^2 + \dot{g}^2 = 1 \Rightarrow \dot{g} = \pm\sqrt{1-a^2}$

$(|a| \leq 1) \Rightarrow g(u) = \pm\sqrt{1-a^2}\, u + c$ — (1)

By applying translation along z-axis

assume $c = 0$.

By applying rotation about x-axis

assume $g = \sqrt{1-a^2}\, u$.

Thus $\sigma(u,v) = (b\cos v, b\sin v, 0) + u(a\cos v, a\sin v, \sqrt{1-a^2})$

— (ruled surface) — (1)

$a = 0$ — ~~Great~~ circular cylinder

$|a| = 1$ — xy-plane

$0 < |a| < 1$ — part of a cone. — (1)

b) $K_G \equiv 1$; $\ddot{f} + f = 0$

$\Rightarrow f(u) = a\cos(u+b)$ — (1)

by reparametrizing $f(u) = a\cos u$

$g(u) = \int \sqrt{1-a^2\sin^2 u}\, du$

— (1)

$a = 0$ — not a surface

$a = 1$ unit sphere

$a = -1$ — " " — (1)

Qn 5.

Jacobian of the transformation

$$J = \begin{pmatrix} \frac{\partial v}{\partial U} & \frac{\partial v}{\partial V} \\ \frac{\partial w}{\partial U} & \frac{\partial w}{\partial V} \end{pmatrix}$$

$$= \begin{pmatrix} v(w+1) & \frac{1}{2}\left(v^2 - (w+1)^2\right) \\ -\frac{1}{2}\left(v^2 - (w+1)^2\right) & v(w+1) \end{pmatrix} \quad - (3)$$

$$FFF = J^t \, FFF(\text{old}) \, J$$

$$= \frac{1}{(1-U^2-V^2)^2} I \quad \left(\text{in matrix form } \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix}\right)$$

$- (3)$