Curves and Surfaces: Solutions to Assignment 1

Qn 1.

Arc length \( s(t) = \int_{a}^{t} \| \dot{\gamma}(u) \| \, du \) is independent of parametrization: use the change of variable \( t \rightarrow \varphi(t) \) \( \text{[3]} \).

\( s \) is a smooth function of \( t \): \( \frac{ds}{dt} = \| \dot{\gamma}(t) \| \text{ [2]} \).

Qn 2.

Let \( \widetilde{\gamma}(t) = \gamma(\varphi(t)) \) where \( \varphi \) is the reparametrization. Let \( \gamma = \varphi^{-1} \) so that \( t = \varphi(t) \).

\( \varphi(\varphi(t)) = t \)

differentiate both sides with \( t \Rightarrow \frac{d\varphi}{dt} \frac{dt}{d\varphi} = 1 \)

\( \Rightarrow \frac{d\varphi}{dt} = \frac{1}{\frac{dt}{d\varphi}} \neq 0 \text{ or } \tilde{\varphi} \).

Next: \( \frac{d\widetilde{\gamma}}{dt} = \frac{d\gamma}{dt} \frac{dt}{d\varphi} \)

This \( \frac{d\widetilde{\gamma}}{dt} \) is never zero if \( \frac{d\gamma}{dt} \) is never zero. \( \text{[2]} \).
\[ \mathcal{X} : (-1,1) \to \mathbb{R}^2, \quad x(t) = (t^3, t^6) \]

\[ \dot{x}(t) = (3t^2, 6t^5) \quad \dot{x}(0) = 0 \]

So \( \dot{x} \) is not regular. By theorem done in lecture, \( \mathcal{X} \) does not have unit speed reparametrization. \[ \text{[3]} \]

Unit speed reparametrization of \((x-x_0) + (y-y_0) = R^2\)

is \( \dot{x}(s) = (x_0 + R\cos \frac{\alpha}{R}, y_0 + R\sin \frac{\alpha}{R}) \)

\[ \text{[12]} \]

**Qn 4**

(a) Recall: If two unit speed plane curves has same curvature then one may be obtained from other by a rigid motion of \( \mathbb{R}^2 \).

If \( K_1 = K_2 \), we have the circle

\[ \mathcal{X}(s) = (\cos \frac{\alpha}{K_2}, \sin \frac{\alpha}{K_2}) \]  

curvature is \( K_2 \).

Any rigid motion takes a circle to circle.

\[ \text{[2]} \]

(b) \( K(x) = K \). Start at \( \dot{x}(0) = 0 \)

\[ t(s) = \int_0^s |\dot{x}(u)| \, du = s^{1/2} \]

\[ x(s) = \left( \int_0^s \cos \frac{s^{1/2} \, dt}{\sqrt{2}}, \int_0^s \sin \frac{s^{1/2} \, dt}{\sqrt{2}} \right) \]

Cornue's spiral

Can you plot it numerically? \[ \text{[2]} \]
(a) The circle passes through \( z(3) \) because
\[ \| z - z_1 \| = \| \frac{1}{k_5} n_5 \| = \frac{1}{|k_5|} \] which is the radius of the circle.
\[ E - z = \frac{1}{k_5} n_5 \] is perpendicular to the tangent \( T \) at \( z \). Hence the circle is tangent to \( T \).
The curvature of the circle is \( \frac{1}{\text{Radius}} \), hence
\[ k_5 - \text{Curvature of } T \]. \[ \frac{1}{k_5} \]

(b) \( E(t) = t + \frac{1}{k_5} (-k_5 t) = \frac{k_5}{k_5^2} n_5 \), \( n_5 = -\frac{k_5}{k_5^2} n_5 \)
arc length of \( E(t) = u_0 - \frac{1}{k_5} \) for some constant \( u_0 \).
Unit tangent vector of \( E \) is \( -n_5 \) and signed unit normal is \( \frac{T}{k} \).
now \( -\frac{dn_5}{du} = \frac{k_5}{k_5} T \) \( \Rightarrow \) signed curvature
of \( E \) is \( k_5^3 / k_5 \).
The free part of the string is long out to \( \gamma \) at \( \gamma(x) \) and has length \( l-d \). Hence the curve traced out is

\[
\gamma(x) = (l-x) \gamma(x)
\]

In the text, see any books referred for the course.

On 8, we need to describe curves up to rigid motions in \( \mathbb{R}^3 \).

Put

\[
a = \frac{k}{k^2 + z^2}, \quad b = \frac{z}{k^2 + z^2}
\]

Circular helix \( \gamma(\theta) = (a \cos \theta, a \sin \theta, b \theta) \) has curvature \( k \) and torsion \( b \).