Curves and Surfaces : Solutions to Assignment-1

Qn 1.

Arc length $s(t) = \int_a^t \| \dot{x}(u) \| \, du$ is independent of parametrization: use the change of variable $t \longmapsto \phi(\tilde{t})$. — [3]

s is a smooth function of t: $\frac{ds}{dt} = \| \dot{x}(t) \|$ [2].

Qn. 2. Let $\tilde{x}(\tilde{t}) = x(\phi(\tilde{t}))$ where ϕ is the reparametrization. Let $\psi = \phi^{-1}$ so that — [1]

$\tilde{t} = \psi(t)$. — [1]

$\phi(\psi(t)) = t$

differentiate both side w.r.t. $t \Rightarrow$ — [1]

$$\frac{d\phi}{d\tilde{t}} \frac{d\psi}{dt} = 1$$

$$\Rightarrow \frac{d\phi}{d\tilde{t}} \neq 0 \quad \forall \tilde{t}.$$

Next. $\frac{d\tilde{x}}{d\tilde{t}} = \frac{dx}{dt} \frac{d\phi}{d\tilde{t}}$

Thus $\frac{d\tilde{x}}{d\tilde{t}}$ is never zero if $\frac{dx}{dt}$ is never zero. — [2].

Qn 3. $\gamma : (-1,1) \to \mathbb{R}^2$, $\gamma(t) = (t^3, t^6)$

$\dot\gamma(t) = (3t^2, 6t^5)$ $\quad \dot\gamma(0) = 0$

So γ is NOT regular. By Theorem done in Lecture γ does NOT have unit speed reparametrization — [3]

Unit speed reparametrization of $(x-x_0)^2 + (y-y_0)^2 = R^2$ is $\gamma(s) = \left(x_0 + R\cos\frac{s}{R},\ y_0 + R\sin\frac{s}{R}\right)$

— [2]

Qn. 4

(a) Recall: If two unit speed planar curves has same curvature then one may be obtained from other by a rigid motion of $\mathbb{R}^2$. — [1]

If $k_s = k$, we have the circle $\gamma(s) = (\cos sk, \sin sk)$ whose curvature is k. — [2]

Any rigid motion takes a circle to circle. — [2]

(b) $k(s) = s$. start at $s_0 = 0$ — [1]

$\varphi(s) = \int_0^s u\, du = s^2/2$ — [2]

$\gamma(s) = \left(\int_0^s \cos t^2/2\, dt,\ \int_0^s \sin t^2/2\, dt\right)$ — [2]

— Cornu's spiral

Can you plot it numerically — ?

Qn 5.

(a) The circle passes through $\gamma(s)$ because
$\| \varepsilon - \gamma \| = \| \frac{1}{k_s} n_s \| = \frac{1}{|k_s|}$ which is the radius of the circle. — [2]

$\varepsilon - \gamma = \frac{1}{k_s} n_s$ is perpendicular to the tangent $\vec{t}$ at γ. Hence the circle is tangent to γ. [2]

The curvature of the circle is $\frac{1}{\text{Radius}}$, hence $|k_s|$ — curvature of γ. [1].

(b) $\dot{\varepsilon}(s) = \vec{t} + \frac{1}{k_s}(-k_s t) - \frac{\dot{k}_s}{k_s^2} n_s = -\frac{\dot{k}_s}{k_s^2} n_s$ — [2]

arc length of $\varepsilon(s) = u_0 - \frac{1}{k_s}$ for some constant u_0.

Unit tangent vector of ε is $-n_s$ and signed unit normal is $\vec{t}$. — [1]

now $-\frac{dn_s}{du} = \frac{k_s^2}{\dot{k}_s}\vec{t} \Rightarrow$ signed curvature of ε is $k_s^3 / \dot{k}_s$. — [2].

Qn 6. The free part of the string is tangent to γ at $\gamma(s)$ and has length $l-s$ — [5].

Hence the curve trace out is

$$\gamma(s) + (l-s)\,\dot{\gamma}(s) \qquad — [5].$$

Qn 7. See any books referred for the course. — [5]

Qn 8. we need to describe curves upto rigid motions in $\mathbb{R}^3$. — [2]

put $a = \dfrac{k}{k^2+\tau^2}$, $b = \dfrac{\tau}{k^2+\tau^2}$.

Circular helix $\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$ has curvature k and torsion b. — [8]