Assignment-1

1. Find maxima and minima points of these functions:

   (a) $f(x) = x^2 + 1$, $x \in (-\infty, \infty)$

   (b) $f(x) = sinx$, $[0, 2\pi]$

   (c) $f(x) = x^2 + 1$, $x \in [-2, 2]$

   (d) $f(x) = 1/x$, $x \in [-1, 1]$

   (e) $f(x) = \begin{cases} 3x, & \text{if } 0 \leq x \leq 1 \\ -x + 4, & \text{if } 1 \leq x \leq 2 \\ 2x - 2, & \text{if } 2 \leq x \leq 3 \end{cases}$

   (f) $f(x) = x^3 - 3x^2$, $x \in \mathbb{R}$

2. Check these sets are convex or non-convex:

   (a) $X = \{x : Ax = b,\ A \in M_{m \times n},\ x \in \mathbb{R}^n,\ b \in \mathbb{R}^m\}$

   (b) $X = \{x : \| x - x_c \| \in r, x \in \mathbb{R}^n, x_c \in \mathbb{R}^n, r \in \mathbb{R}\}$

   (c) $X = \{(x, y) : y \geq -x^2, x \in \mathbb{R}\}$

   (d) $X = \{1, 2, 3, ...\}$

3. Check weather these functions are convex or not:

   (a) $f(x) = ax + b,\ a, b, x \in \mathbb{R}$

   (b) $f(x) = e^{ax},\ a, x \in \mathbb{R}$

   (c) $f(x) = | x |^p,\ x \in \mathbb{R},\ p \geq 1$

   (d) $f(x) = -x^2,\ x \in \mathbb{R}$

   (e) $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$

4. Find closure and interior of these sets:

   (a) $A = [0, 1]$

   (b) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

   (c) $A = \{ax + b : a, b, x \in \mathbb{R}\}$

   (d) $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$

5. Find convex hull of these sets:

   (a) $A = [0, 1] \subseteq \mathbb{R}$
   (b) $A = [0, 1] \bigcup \{2\} \subseteq \mathbb{R}$
   (c) $A = \{(0, 0)\} \bigcup \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$
   (d) $A = \{(0, 0)\} \bigcup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

6. True and False:

   (a) Feasible set of linear programming problem is always polyhedral.
   (b) A function is improper if it is not proper function.
   (c) If C is compact then support function σ_C may take the value $+\infty$.
   (d) If C is not compact then support function σ_C may take the value $+\infty$.

7. True and False:

   (a) $A = \{(x, y) : x > 0, y > 0\} \bigcup \{(x, y) : x < 0, y < 0\}$ *and* $B = \{(x, y) : x < 0, y > 0\} \bigcup \{(x, y) : x > 0, y < 0\}$ can be separated.
   (b) $A = \{1\}$ *and* $B = (1, 2]$ can be strictly separated.
   (c) Strongly convex function may have more than one minimizer over a closed convex set.
   (d) Distance function is convex function.
   (e) C_1-convex and compact, C_2-convex and closed with $C_1 \bigcap C_2 = \emptyset$.Then strict separation is possible.
   (f) If we assume closedness in place of compactness of C_1 then also strict separation is possible.
   (g) If $C_1 = \textit{epi graph of } 1/x, x > 0$ and $C_2 = \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$ then strict separation is not possible.
   (h) If f is strictly convex then minimizer of f(x) is unique.

8. True and False:

   (a) Every convex function is continuous.
   (b) $f : \mathbb{R}^n \to \mathbb{R}$, f is convex then f may or may not be continuous.

(c) $f : \mathbb{R}^n \to \mathbb{R}$, f is convex differentiable then ∇f has monotonocity property.

(d) $f : C \to \mathbb{R}$, C-closed convex set then f may or may not be continous.

(e) $f(x) =| x |$ is differentiable function.

(f) $f(x) = x^3 + x$ is convex function.

9. Write down argmin of given functions:

(a) $f(x) = x^2, \ x \in \mathbb{R}$

(b) $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

10. True and False:

(a) If $f(x) = |x|$ then $\partial f(0) = [-1, 1]$.

(b) Necessary and sufficient condition for optimality of local minima $\bar{x}$ is $0 \in \partial f(\bar{x})$

(c) $f : \mathbb{R}^n \to \mathbb{R}$ convex then $\partial f(x)$ can be empty for some function.

(d) If f is differentiable then $\partial f(x)$ is singleton.

(e) $\partial f(x)$ is convex and compact set for $x \in dom(f)$.

(f) If $f : \mathbb{R}^n \to \mathbb{R}$ and f is convex then f is locally lipschitz.

(g) If $f : \mathbb{R}^n \to \mathbb{R}$ and locally lipschitz then f is convex.

(h) $f^{'}(x, h) = \max\limits_{\xi \in \partial f(x)} \langle \xi, h \rangle$.

11. Find directional derivatives of these functions:

(a) $f(x) = |x|, \ x \in \mathbb{R}, \ find \ f^{'}(0, v), \ v \in \mathbb{R}$

(b) $f(x) = \langle x, b \rangle$,b-fix, $x, b \in \mathbb{R}^2$, find $f^{'}(x, v), \ v \in \mathbb{R}, x = (1, 1)$

12. True and False:

(a) $\partial(f_1 + f_2)(x) \subseteq \partial f_1(x) + \partial f_2(x)$ but reverse inclusion does not hold.

(b) $\partial(\lambda f)(x) = \lambda \partial f(x)$ only for $\lambda > 0$

(c) If $f(x) = \max\{f_1(x), f_2(x), ..., f_n(x)\}$ then $\partial f(x) = conv\{\nabla f_i(x) : i \in J(x)\}, \ where \ J(x) = \{i \in \{1, 2.., m\} : f_i(x) = f(x)\}$

(d) If x is point of f such that f is not finite.then $\partial f(x)$ can be non empty.

(e) $\partial \delta_C(x) = \{v : \langle v, y - x\rangle \leq 0, \ \forall y \in C\}$ where C is convex set.

(f) $\bar{x}$ is minimizer of $f(x)$ iff $\bar{x}$ is also a minimizer of the problem $(f + \delta_C)(x)$

13. Find normal cone of these sets:

(a) $C = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$, find $N_C(x_0)$, where $\|x_0\| = 1$

(b) $C = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ find $N_C((1, 1))$

(c) $C = [0, 1], \ find \ N_C(1)$

(d) $C = \{(x, y) : y = 0\}, \ find \ N_C((0, 0))$

14. True and False

(a) $\delta_{C_1 \bigcap C_2}(x) = \delta_{C_1}(x) + \delta_{C_2}(x)$

(b) $N_{C_1 \bigcap C_2}(x) \neq N_{C_1}(x) + N_{C_2}(x)$

(c) If C=$\{x : Ax = b\}$ then $N_C(\bar{(x)}) = \mathrm{ImA^T}$, where $\bar{x}$ is solution of the problem $\min\limits_{x \in C} f(x)$

15. Check weather these functions are lower semi-continuous function:

(a) f(x)=$\begin{cases} 1, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ 1/2, & \text{if } x > 1 \end{cases}$

(b) f(x)=$\begin{cases} 1/x, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$

(c) f(x)=$\begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$

(d) f(x)=$\begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$

16. True and False:

(a) Polyhedral set is intersection of infinite number of closed half spaces.

(b) $\mathbb{R}_n^+$ is polyhedral set.

(c) Polyhedral cone is finitely generated.

(d) A convex cone which is polyhedral may have infinite number of generators.

17. True and False:

(a) If $\bar{x}$ is minimizer for f(x) over C then $f(\bar{x}+\lambda w)-f(\bar{x}) \geq 0,\ \forall\, w \in T_C(x),\ \lambda > 0$.

(b) For cone K, $(K^0)^0 = K$.

(c) If K is closed convex cone then $(K^0)^0 = K$.

(d) $T_C(\bar{x})^0 = N_C(\bar{x})$.

(e) $N_C(\bar{x})^0 \neq T_C(\bar{x})$.

(f) If f is proper convex function the f^* not always proper function.

18. Fill the blanks:

(a) If $f(x) = \frac{x^2}{2},\ x \in \mathbb{R}$, then $f^*(x^*) =$ ....

(b) If $f(x) = \frac{\|x\|^2}{2},\ x \in \mathbb{R}$, then $f^*(x^*) =$ ....

(c) $f(x) = \delta_C(x),\ C - \textit{closed set}$, then $f^*(x^*) =$ ....

19. True and False:

(a) Behind every minimization problem there is a maximization problem.

(b) Val(CP) is always equal to Val(DP).

(c) If slater condition hold then Val(CP)= Val(DP).

20. Fill the blanks:

(a) Bounded polyhedral sets are called ....

(b) Extreme points of a polyhedral set are ....

(c) Extreme points are in the .... of sets.

(d) x is extreme point then if $x = \frac{x_1+x_2}{2}$ then x=.......