Assignment - 9

1. a) $E(N(t)-t) = E(N(t)) - t = t - t = 0 < \infty$

b) $N(t)$ is adapted w.r.t. natural filtration

$\therefore N(t)-t$ is adapted w.r.t natural filtration.

c) $E(N(t)-t \mid F(s)) = E(N(t) \mid F_s) - t$

$= E(N(t) - N(s) + N(s) \mid F_s) - t$

$= E(N(t) - N(s) \mid F_s) + N(s) - t$

$= E(N(t) - N(s)) + N(s) - t$

$= t - s + N(s) - t$

$= N(s) - s$

$\therefore \{N(t) - t \,;\, t \geq 0\}$ is a martingale

Hence option (a) is correct.

2. $X(t) = e^{W(t) - ct}, \quad t \geq 0$

Since $X(t)$ is a Martingale.

$E(X(t) \mid F_s) = X(s)$

$E\left(e^{W(t) - ct} \mid F_s\right) = e^{W(s) - cs}$

$e^{-ct} e^{W(s)} E\left(e^{W(t) - W(s)} \mid F_s\right) = e^{W(s) - cs}$

$e^{W(s) - ct} \cdot e^{\frac{t-s}{2}} = e^{W(s) - cs}$

$-ct + \frac{t-s}{2} = -cs$

$c(t-s) = \frac{(t-s)}{2}$

$c = 1/2$ option (b).

(3) $S(t) = \mu t + \sigma W(t)$

i) $E(S(t)) = \mu t + \sigma E(W(t)) = \mu t < \infty$

ii) $W(t)$ is adapted w.r.t F_t

$\Rightarrow S(t)$ is adapted w.r.t F_t.

iii) $E(S(t) | F_s) = E(\mu t + \sigma W(t) | F_s)$

$s \le t$ $= \mu t + \sigma E(W(t) | F_s)$

$= \mu t + \sigma E(W(t) - W(s) + W(s) | F_s)$

$= \mu t + \sigma E(W(t) - W(s)) + \sigma W(s)$

$= \mu t + \sigma W(s)$

$= (\mu s + \sigma W(s)) + \mu(t-s)$

$\therefore$ it is sub-martingale if $\mu > 0$

it is super-martingale if $\mu < 0$

it is Martingale if $\mu = 0$

option (a) is answer.

(6) $S(t) = c(\sigma+1)^{N(t)}$

a) $E(S(t)) = c\,E((\sigma+1)^{N(t)})$

$= c\, e^{\lambda t(\sigma+1-1)}$ $\left[\because X \sim P(\lambda),\ E(s^X) = e^{\lambda(s-1)}\right]$

$= c\, e^{\sigma \lambda t} < \infty$

b) $S(t)$ is adapted w.r.t. filtration

c) $E\left(c(\sigma+1)^{N(t)} \mid F_s\right)$

$= E\left(c(\sigma+1)^{N(t)-N(s)+N(s)} \mid F_s\right)$

$= c(\sigma+1)^{N(s)} E\left((\sigma+1)^{N(t)-N(s)} \mid F_s\right)$

$= c(\sigma+1)^{N(s)} E\left((\sigma+1)^{N(t)-N(s)}\right)$

$= c(\sigma+1)^{N(s)} e^{d(t-s)(\sigma+1-1)} = c(\sigma+1)^{N(s)} e^{\sigma d(t-s)}$

Since, it is Martingale

$c(\sigma+1)^{N(s)} e^{d\sigma(t-s)} = c(\sigma+1)^{N(s)}$

If we chose $c = e^{-\sigma d t}$

then $E\left(e^{-\sigma d t} \cdot (\sigma+1)^{N(t)} \mid F_s\right)$

$= e^{-d\sigma s} (\sigma+1)^{N(s)}$

$\therefore \boxed{c = e^{-\sigma d t}}$ option (a)

8. $\cancel{y_n = (-1)^n \cos(\pi X_n)}$

i) $\cancel{E(y_n) = (-1)^n E(\cos(\pi X_n)) < \infty}$

$G_1 \subseteq G_2$

$E(X \mid G_1) = E\left(E(X \mid G_2) \mid G_1\right)$

option $\underline{(b)}$. correct.

(9). If $\mathbb{F} = \{\phi, \Omega\}$ the only possible random variable is a constant.

Hence, $E(X|F) = K$ where $P(X=K) = 1$

option (d) is correct.

(7). $X(t) = e^{\sigma W(t) - \frac{\sigma^2}{2}t}$

i) $E(X(t)) = E(e^{\sigma W(t) - \frac{\sigma^2}{2}t})$

$= e^{-\frac{\sigma^2}{2}t} E(e^{\sigma W(t)})$

$= e^{-\frac{\sigma^2}{2}t} \cdot e^{\frac{\sigma^2 t}{2}} = e^0 = 1 < \infty$

ii) Clearly, $X(t)$ is adapted w.r.t. filtration generated by $W(t)$.

iii) $E(X(t)|F_s) = E(e^{\sigma W(t) - \frac{\sigma^2}{2}t} | F_s)$

$= e^{-\frac{\sigma^2}{2}t} E(e^{\sigma W(t)} | F_s)$

$= e^{-\frac{\sigma^2}{2}t} E(e^{\sigma(W(t)-W(s)) + \sigma W(s)} / F_s)$

$= e^{-\frac{\sigma^2}{2}t} e^{\sigma W(s)} E(e^{\sigma(W(t)-W(s))} / F_s)$

$= e^{\sigma W(s) - \frac{\sigma^2}{2}t} E(e^{\sigma(W(t)-W(s))})$

[$\because W(t) - W(s)$ is independent of F_s]

$$= e^{\sigma W(s) - \frac{\sigma^2}{2}t} \cdot e^{\frac{\sigma^2}{2}(t-s)}$$

$$= e^{\sigma W(s) - \frac{\sigma^2}{2}s} = X(s)$$

$\therefore$ $X(t)$ is a martingale

option (a) correct

(4) A stochastic process $\{X_t\}_{t \geq 0}$ is a Martingale w.r.t. filtration $\{F_t\}_{t \geq 0}$ if

a) $E(X_t)$ exists

b) X_t is measurable w.r.t. F_t

c) $E(X_t | F_s) = X_s$ where $s \leq t$

option (a) is correct.

(5) Martingale : if (a) (b) (c) are satisfied from ques. (4) conditions.

Submartingale if (a), (b) & $E(X_t | F_s) \geq X_s$ when $s \leq t$

Supermartingale if (a), (b) & $E(X_t | F_s) \leq X_s$ when $s \leq t$

Hence option (a), (b) & (c) are correct.

Ans. (d). (which is FALSE)