(2) $X \sim \exp(\lambda)$

$$p_j = (1-\alpha)\alpha^j = P(j \le X < j+1)$$

$$= \int_j^{j+1} f(x)\,dx$$

$$= \int_j^{j+1} \lambda e^{-\lambda x}\,dx = (1-e^{-\lambda})(e^{-\lambda j})$$

$$\Rightarrow (1-\alpha)\alpha^j = (1-e^{-\lambda})(e^{-\lambda j})$$

$$\Rightarrow \boxed{\alpha = e^{-\lambda}}$$

(1) $Y = a - bX$ where $X \sim N(\mu, \sigma^2)$

Let $b > 0$ $\quad P(Y \le y) = P(a - bX \le y)$

$$= P(-bX \le y-a)$$

$$= P\left(X \ge \frac{y-a}{-b}\right)$$

$$= 1 - \Phi\left(\frac{y-a}{-b}\right)$$

$\therefore$ pdf of y is

$$g(y) = \frac{1}{b}\phi\left(\frac{y-a}{b}\right)$$

where Φ is CDF of $N(\mu, \sigma^2)$

ϕ is pdf of $N(\mu, \sigma^2)$

$$\Rightarrow Y \sim N(a - b\mu, b^2\sigma^2)$$

Assignment (Week 2)

(1) $X \sim N(\mu, \sigma^2)$

$Y = a - bX = g(X)$

$\frac{dy}{dx} = -b$ which is constant, so keeps same sign for all x.

$\therefore$ pdf of Y is

$g(y) = \begin{cases} f(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| , & -\infty < -|a - bx < \end{cases}$

Similarly, when $b < 0$

$P(Y \le y) = P(a - bX \le y)$

$= P(-bX \le y - a)$

$= P\left(X \le \frac{y-a}{-b}\right)$

$= \Phi\left(\frac{y-a}{-b}\right)$

$\therefore$ pdf of Y is $\frac{1}{-b}\phi\left(\frac{y-a}{-b}\right)$ $\cancel{N}$

i.e. $Y \sim N(a - b\mu, b^2\sigma^2)$

Hence, $\boxed{Y \sim N(a - b\mu, b^2\sigma^2)}$

(3) $Y = F_X(x)$

$$P(Y \le y) = P(F_X(x) \le y)$$
$$= P(X \le F_X^{-1}(y))$$
$$= F_X(F_X^{-1}(y)) = y \quad \text{when } 0 \le y < 1$$

$$\therefore P(Y \le y) = \begin{cases} 0 & \text{if } \cancel{\infty <} y < 0 \\ y & \text{if } 0 \le y < 1 \\ 1 & \text{if } 1 \le y \end{cases}$$

$\therefore Y \sim U(0,1)$.

(4) $E(X) = 1, \quad E(X^2) = 1$

$\Rightarrow \quad V(X) = 0$

$\therefore X$ is a degenerate (constant) random variable.

& $P(X = 1) = 1$

$\therefore \quad P(-\frac{1}{2} < X \le 3) = P(X = 1) = 1$

(5) $M_X(t) = \frac{1}{6} + \frac{1}{2} e^{-t} + \frac{1}{3} e^{t}$

By uniqueness of MGF, we have
X is a discrete random variable with possible values $\{0, -1, 1\}$

$$P(X = -1) = \frac{1}{2}, \quad P(X = 0) = \frac{1}{6}, \quad P(X = 1) = \frac{1}{3}$$

$$E(X) = (-1)\frac{1}{2} + 0\left(\frac{1}{6}\right) + 1\left(\frac{1}{3}\right) = -\frac{1}{2} + \frac{1}{3} = \frac{-1}{6}$$

$$E(x^2) = (-1)^2 \frac{1}{2} + (0)^2 \frac{1}{6} + (1)^2 \frac{1}{3}$$

$$= \frac{1}{2} + \frac{1}{3} = 5/6$$

$$\therefore \sigma = \sqrt{\frac{5}{6} - \left(\frac{-1}{6}\right)^2} = \sqrt{\frac{5}{6} - \frac{1}{36}} = \frac{\sqrt{29}}{6}$$

$$P(\mu - \sigma < X < \mu + \sigma) = P(-1.06 < X < 0.73)$$

$$= P(X = -1) + P(X = 0)$$

$$= \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

(6)

$$\sum_{y=0}^{\infty} P(X = y) = 1 \qquad [\because \text{pmf}]$$

$$\Rightarrow \sum_{y=0}^{\infty} (1-k) k^y = 1$$

$$\Rightarrow (1-k) + (1-k)k + (1-k)k^2 + \dots = 1$$

which is true if $k < 1$

$\therefore$ option (c) is correct.

~~k≠0~~

(7)

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_0^1 (\alpha + \beta x^2)\, dx = 1$$

$$\Rightarrow \boxed{\alpha + \frac{\beta}{3} = 1} \quad \text{— (1)}$$

$$E(x) = \frac{3}{5}$$

$$\int_0^1 x(\alpha + \beta x^2)\, dx = \frac{3}{5}$$

$$\Rightarrow \boxed{\frac{\alpha}{2} + \frac{\beta}{4} = \frac{3}{5}} \quad \text{— (2)}$$

Solving ① & ②, we have

$$\boxed{\alpha = \frac{3}{5}, \quad \beta = \frac{6}{5}}$$

(8) $X \sim U(0,1)$

$\therefore$ pdf is

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y = -2 \ln X = g(X)$$

$$\frac{dy}{dx} = \frac{-2}{x} < 0 \quad \forall \quad 0 < x < 1$$

$\therefore$ pdf of Y is given by

$$g(y) = \begin{cases} f(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & ; \ 0 < g^{-1}(y) < 1 \\ 0 & ; \ o/w \end{cases}$$

$$= \begin{cases} 1 \times \left| \frac{d}{dy} e^{-y/2} \right| & ; \ 0 < e^{-y/2} < 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-y/2} & ; \ y > 0 \\ 0 & ; \ \text{otherwise} \end{cases}$$

$\therefore$ Y follows exponential distribution.