(1) $P(\text{First arrival is a girl}) = P(X_2 < X_1)$

where $X_2 \sim \exp(\lambda_2)$

$X_1 \sim \exp(\lambda_1)$

and they are independent

$$P(X_2 < X_1) = \int_0^\infty \int_0^{x_1} f(x_1) f(x_2)\, dx_2\, dx_1$$

$$= \frac{\lambda_2}{\lambda_1 + \lambda_2}$$ option (b)

(2) $P(X_1 > t) = P(N(t) = 0)$ [i.e. no renewal till time t]

$$= e^{-\lambda} \frac{\lambda^0}{0!} = e^{-\lambda}$$

$\Rightarrow P(X_1 > t) = e^{-\lambda}$

$\therefore X_1 \sim \exp(\lambda)$

option (b)

(3) $X_1(t)$ & $X_2(t)$ be two indep. Poisson process with rate $\lambda_1$ & $\lambda_2$.

Then $X_1(t) + X_2(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

option (b)

(4) Mean function of renewal process is $3t$.

We know that $m(t)$ characterizes the renewal process & $m(t)$ is linear for Poisson process

i.e. $X(t) \sim$ Poisson process $(\lambda)$

$$m(t) = \lambda t.$$

$\therefore$ ~~X(t)~~ $X(t)$ is Poisson process with rate 3.

Hence, $P(X(10)=0) = e^{-10\times 3} \frac{(10\times 3)^0}{0!} = e^{-30}$

option (a).

(5) $m(t) = E(N(t)) = \lambda t$ $[\because N(t) \sim \text{Poisson}(\lambda t)]$

$\therefore$ option (b) (linear function of $t$).

(6) Clearly,

$$N(t) \geq n \iff S_n \leq t$$

$[\because N(t) = \max\{n \mid S_n \leq t\}]$

$\therefore$ option (a)

(2)

(7) Inter-arrival distribution is $U(0,1)$

$$m(t) = E(N(t)) = \sum_{n=1}^{\infty} F_n(t)$$

where $F_n(t)$ is $n^{th}$ convolution of $X \sim U(0,1)$

$$F_n(t) = P(S_n \le t)$$

$$= \int_0^t P(t_n \le t - x) \frac{x^{n-2}}{(n-2)!} dx$$

$$= \frac{t^n}{n!} \; ; \; 0 \le t \le 1$$

$$\therefore m(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} = e^t - 1 \; ; \; t \in [0,1]$$

$\therefore$ option (c)

(8)

Class notes

Longrun average cost $= \dfrac{E(R)}{E(X)}$.

option (d)

(9) Expected life time of a battery

$$= E(X) = \frac{1+25}{2} = 13$$

Hence, the rate at which photographer has to change the battery is

$$\frac{1}{E(x)}$$

$$= \frac{1}{13}$$

option (a)

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(10) Let $f(x) = \alpha^2 x e^{-\alpha x}$ —— (1)

renewal density

$$m(t) = f(t) + \int_0^t m(t-u) f(u)\, du$$

Taking Laplace

$$\bar{m}(s) = \frac{\bar{f}(s)}{1 - \bar{f}(s)} \quad \text{—— (2)}$$

from (1) $\bar{f}(s) = L(\alpha^2 x e^{-\alpha x}) = \alpha^2 L(e^{-\alpha x} \cdot x)$

$$= \frac{\alpha^2}{(s+\alpha)^2}$$

$$\bar{m}(s) = \frac{\frac{\lambda^2}{(s+\lambda)^2}}{1 - \frac{\lambda^2}{(s+\lambda)^2}} = \frac{\lambda^2}{(s+\lambda)^2 - \lambda^2} = \frac{\lambda^2}{s^2 + 2\lambda s} = \frac{\lambda^2}{s(s+2\lambda)}$$

Taking Laplace inverse

$$m(x) = L^{-1}\left(\frac{\lambda^2}{s(s+2\lambda)}\right)$$

$$= \frac{\lambda}{2} L^{-1}\left(\frac{1}{s} - \frac{1}{s+2\lambda}\right)$$

$$m(x) = \frac{\lambda}{2}\left[1 - e^{-2\lambda x}\right]$$

$$\therefore \quad M(x) = \int_0^x m(t)\,dt = \frac{\lambda}{2}x - \frac{\lambda}{2}\left[\frac{e^{-2\lambda t}}{-2\lambda}\right]_0^x$$

$$= \frac{\lambda}{2}x + \frac{1}{4}\left[1 - e^{-2\lambda x}\right]$$

$\therefore$ option (d)