Assignment - 10

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(1) Let $s \leq t$

$Cov(W(t), W(s))$

$= E(W(t)W(s)) - E(W(t))E(W(s))$

$= E(W(t)W(s)) - 0 \times 0$

$= E((W(t) - W(s) + W(s))W(s))$

$= E(W(s)(W(t) - W(s))) + E((W(s))^2)$

$= E(W(s))E(W(t) - W(s)) + s$ [$\because$ $W(s)$ & $W(t) - W(s)$ are independent]

$= 0 \times 0 + s$

$= s$

Similarly, if $t \leq s$

$Cov(W(t), W(s)) = t$

$\Rightarrow Cov(W(s), W(s)) = \min\{s, t\}$

$\therefore$ option (a)

(2) $E(S(t)) = S(0) e^{t(\mu + \sigma^2/2)}$ when $S(t)$ is a geometric Brownian motion.

Here, $S(0) = 100$, $\mu = 0.01$, $\sigma = 0.2$

$E(S(10)) = 100 e^{10(0.01 + \frac{(0.2)^2}{2})}$

$= 100 e^{0.3} = \underline{\underline{134.98}}$ Ans

(3) Geometric B.M is given by

$$S(t) = S(0)\, e^{\mu t + \sigma W(t)}$$

& $E(S(t)) = S(0)\, e^{\mu t + \frac{\sigma^2 t}{2}}$

∴ when $\mu = 0.01$, $\sigma = 0.2$, $S(0) = 100$

$$E(S(10)) = 100\, e^{(0.01)(10) + \frac{(0.2)^2 \times 10}{2}}$$

$$= 100\, e^{0.1 + 0.2}$$

$$= 100\, e^{0.3}$$

(3) We know that $W(t) - W(s) \sim N(0, t-s)$

$$\therefore E\left((W(t) - W(s))^2\right) = \cancel{3(t-s)}$$

$= E(X^2)$ where $X \sim N(0, t-s)$

$= V(X) + (E(X))^2$

$= t - s$

∴ option (c)

(4) $E(S(t)) = S(0)\, e^{t(\mu + \sigma^2/2)} = a e^{\mu t + \frac{\sigma^2 t}{2}}$

$S(t) = S(0)\, e^{\mu t + \sigma W(t)}$

while $\cancel{E(S(t)) = S(0)}$

$\left[\because E(e^{\sigma W(t)}) = e^{\sigma^2 t/2}\right]$

∴ option (a)

(5) $E\left(W(s) \,\middle|\, W(t)=x\right)$ where $s<t$ (2) — (1)

(2)

We know by time inversion property that

for $t>0$ $\tilde{W}(t) = \{t\,W(\tfrac{1}{t})\}$ is a Weiner process

$\tilde{W}(0)=0$

$\therefore$ Condition ~~distribution~~ expectation of $W(s)$ given $W(t)=x$ is same as conditional ~~distribution~~ expection of $sW(\tfrac{1}{s})$ given $tW(\tfrac{1}{t})=x$.

$$\therefore E\left(s\,W(\tfrac{1}{s}) \,\middle|\, t\,W(\tfrac{1}{t})=x\right) = s\,E\left[W(\tfrac{1}{s})-W(\tfrac{1}{t})+W(\tfrac{1}{t}) \,\middle|\, tW(\tfrac{1}{t})=x\right]$$

$$= s\left[E(W(\tfrac{1}{s})-W(\tfrac{1}{t})) + \frac{x}{t}\right]$$

$$= s\left[0+\frac{x}{t}\right] = \frac{sx}{t}$$

$\therefore$ option (a).

(6) From Que 1, we have

$$\mathrm{Cov}(W(s), W(t)) = \min\{s,t\}$$

$\therefore$ when $0<s<t$

$$\mathrm{Cov}(W(s), W(t)) = s$$

$\therefore$ option (a) is correct.

(7) ~~From Assignment 9, Que~~

(a) $E((W(t))^2) = t < \infty$

(b) $(W(t))^2$ is adapted wrt $\mathcal{F}_t$

c) $E((W(t))^2 / F_s) = E((W(t) - W(s) + W(s))^2 / F_s)$

$= E((W(t) - W(s))^2 / F_s) + E((W(s))^2 / F_s)$

$+ 2E(W(s)W(t) - W(s)) / F_s)$

$= E((W(t) - W(s))^2 / F_s) + (W(s))^2 + 2W(s) E(W(t) - W(s))$

$= t - s + W(s)^2 + 0$

$= W(s)^2 + t - s \geq (W(s))^2$

$\therefore$ gt is a sub. martingale

option (b)

8) $E(W(t) | W(s) = x)$ when $s < t$

$= E(W(t) - W(s) + W(s) | W(s) = x)$

$= E(W(t) - W(s) | W(s) = x) + E(W(s) | W(s) = x)$

$= E(W(t) - W(s)) + x$ [$\because$ $W(t) - W(s)$ is independent of $W(s)$]

$= 0 + x$

$= x$

$\therefore$ option (d)