Stochastic Processes
Assignment (Week 9)

1. Let \( \{N(t), t \geq 0\} \) be a Poisson process with rate 1. Which of the following stochastic process is martingale with respect to the natural filtration?

   (a) \( \{N(t) - t, t \geq 0\} \)
   (b) \( \{N(t)^2 - t, t \geq 0\} \)
   (c) \( \{(N(t) - t)^2 - t, t \geq 0\} \)
   (d) \( \{N(t), t \geq 0\} \)

2. Which of the following statements is TRUE?

   (a) The mean of a martingale process is constant with respect to time.
   (b) The mean of a martingale process is time varying.
   (c) The mean of a martingale process need not exist.
   (d) Every stochastic process is a martingale.

3. Which of the following statements about a stochastic process \( \{X(t), t \geq 0\} \) is FALSE?

   (a) The process \( \{X(t), t \geq 0\} \) is sub-martingale if and only if the process \( \{-X(t), t \geq 0\} \) is super-martingale.
   (b) The process \( \{X(t), t \geq 0\} \) is a martingale implies that the process \( \{X(t), t \geq 0\} \) is sub-martingale.
   (c) The process \( \{X(t), t \geq 0\} \) is a martingale if and only if the process \( \{X(t), t \geq 0\} \) is both a sub-martingale and a super-martingale.
   (d) The process \( \{X(t), t \geq 0\} \) is a martingale if and only if it is Markov process.

4. If \( \{W(t), t \geq 0\} \) is a Brownian motion, then which of the following is NOT a martingale?

   (a) \( \{W(t)^2 - t, t \geq 0\} \)
   (b) \( \{W(t), t \geq 0\} \)
   (c) \( \{W(t) - t, t \geq 0\} \)
   (d) \( \{W(t)^2, t \geq 0\} \)

5. For what values of \( c \), the process \( S(t) = c(\sigma+1)^{N(t)}, \sigma > -1 \) is a constant, is a martingale where \( \{N(t), t \geq 0\} \) is a Poisson process with rate \( \lambda \).

   (a) \( e^{-\lambda t} \)
   (b) \( e^{\lambda t} \)
   (c) \( \sigma \)
   (d) \( \lambda \)

6. Let \( X_n \) be a symmetric random walk and \( \mathbb{F}_n \) be a filtration. The stochastic process

   \[ Y_n = (-1)^n \cos(\pi X_n) \]

   is

   (a) a martingale with respect to \( \mathbb{F}_n \).
   (b) a sub-martingale with respect to \( \mathbb{F}_n \) but not martingale.
   (c) a super-martingale with respect to \( \mathbb{F}_n \) but not martingale.
   (d) not a martingale with respect to \( \mathbb{F}_n \).
7. Let \( \{W(t), t \geq 0\} \) be a Wiener process. The stochastic process \( \exp(\sigma W(t) - \frac{\sigma^2}{2} t) \), where \( \sigma \) is a positive constant

(a) a martingale with respect to \( \mathbb{F}_n \).
(b) a sub-martingale with respect to \( \mathbb{F}_n \) but not martingale.
(c) a super-martingale with respect to \( \mathbb{F}_n \) but not martingale.
(d) not a martingale with respect to \( \mathbb{F}_n \).

8. Let \( X \) be a random variable and \( G_1 \) and \( G_2 \) be two sub-\( \sigma \)-fields of \( \mathbb{F} \) such that \( G_1 \subseteq G_2 \). Which of the following statements is FALSE?

(a) \( E(E(X \mid G_1) \mid G_2) = E(X \mid G_1) \).
(b) \( E(E(X \mid G_2) \mid G_1) = E(X \mid G_1) \).
(c) \( E(E(X \mid G_1) \mid G_2) = E(X \mid G_2) \).
(d) \( E(E(X \mid G_1) \mid G_2) = E(E(X \mid G_2) \mid G_1) \).

9. Let \( X \) be a random variable and let \( \mathbb{F} = [\emptyset, \Omega] \) be the \( \sigma \)-field. Then,

(a) \( E(X \mid \mathbb{F}) \) need not exist.
(b) \( E(X \mid \mathbb{F}) \) always exists.
(c) \( X \) must be a degenerate random variable.
(d) \( E(X \mid \mathbb{F}) \) always exists and is same as value of \( X \).

10. For what values of \( c \), the process \( S(t) = e^{W(t) - ct} \) is a martingale where \( \{W(t), t \geq 0\} \) is a Wiener process.

(a) \(-0.5\)
(b) \(0.5\)
(c) \(1\)
(d) \(-1\)

11. Consider the process \( S(t) = \mu t + \sigma W(t) \) where \( \{W(t), t \geq 0\} \) is a Wiener process. Which of the following is FALSE?

(a) \( S(t) \) is a martingale for every value of \( \mu \).
(b) \( S(t) \) is a martingale only if \( \mu = 0 \).
(c) \( S(t) \) is a sub-martingale for \( \mu > 0 \).
(d) \( S(t) \) is a super-martingale for \( \mu < 0 \).