Stochastic Process
Assignment (Week 6)

1. Let $\{X(t), t \geq 0\}$ be a continuous time Markov Chain with state space $S = \{0,\ 1,\ 2\}$ and

$$P = \begin{bmatrix} 2 & -1 & 1 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 1 & 2.5 & -3.5 \end{bmatrix},\ Q = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

which of the above mentioned two matrix could be a possible infinitesimal generator matrix for $\{X(t), t \geq 0\}$?
(a) Both P and Q (b) Only P (c) Only Q (d) Neither P nor Q.

2. For an irreducible, positive recurrent continuous-time Markov chain with infinitesimal generator matrix $Q = [q_{ij}]$ and probability vector $\pi = (\pi_0,\ \pi_1, \ldots)$, the time reversibility equation is given by

   (a) $q_{ij} = q_{ji} \quad \forall i, j.$
   (b) $q_{ij} = \frac{\pi_i q_{ji}}{\pi_j} \quad \forall i, j.$
   (c) $q_{ij} = \frac{\pi_j q_{ji}}{\pi_i} \quad \forall i, j.$
   (d) $q_{ij} = \pi_i q_{ji} \quad \forall i, j.$

3. The backward and forward Kolmogorov equations for a Continuous time markov chain $\{X(t),\ t \geq 0\}$ with transition probabilities matrix $P(t)$ and infinitesimal generator matrix Q are given by

   (a) Both the equations are $P'(t) = P(t)Q$.
   (b) $P'(t) = QP(t)$, $P'(t) = P(t)Q$ respectively.
   (c) $P'(t) = P(t)Q$, $P'(t) = QP(t)$ respectively.
   (d) Both the equations are $P'(t) = QP(t)$.

Answer the following 3 questions based on the problem statement given below.
Consider a service station with two identical computers and two technicians. Assume that when both computers are in good condition, most of the work load is on one computer, exposed to a failure rate $\lambda = 1$, while the other computers failure rate is $\lambda = 0.5$. Further assume that, if one of the computer fails, the other one takes the full load, thus exposed to a failure rate $\lambda = 2$. Among the technicians, one is with repair rate $\mu = 2$ while the second is with repair rate $\mu = 1$. If both work simultaneously on the same computer, the total repair rate is $\mu = 2.5$. Note that, at any given moment, they work so that repair rate is maximized.

4. Classify the stochastic process $\{X(t),\ t \in I\}$, denoting the number of computers in good condition at time t based on the state space and time domain.

   (a) Discrete time discrete state stochastic process.
   (b) Discrete time continuous state stochastic process.
   (c) Continuous time discrete state stochastic process.
   (d) Continuous time continuous state stochastic process.

5. Classify the states of the Markov chain $\{X(t),\ t \geq 0\}$ with state space $S = \{0, 1, 2\}$.

   (a) All states are transient.
   (b) All states are recurrent.
   (c) The states 0 and n are absorbing states and all other states are transient.
   (d) All states are transient except state 0, which is an absorbing state.

6. The infinitesimal generator matrix, $Q = [q_{ij}]_{i,j \in S}$ of the Markov chain $\{X(t),\ t \geq 0\}$ with state space $S = \{0, 1, 2, \ldots, n\}$ is given by

(a)

$$Q = \begin{bmatrix} -3 & 3 & 0 \\ 2 & -4.5 & 2.5 \\ 0 & 1.5 & -1.5 \end{bmatrix}$$

(b)

$$Q = \begin{bmatrix} -2.5 & 2.5 & 0 \\ 2 & -4.5 & 2.5 \\ 1.5 & 1 & -2.5 \end{bmatrix}$$

(c)

$$Q = \begin{bmatrix} -3 & 3 & 0 \\ 0.5 & -3 & 2.5 \\ 0 & 1.5 & -1.5 \end{bmatrix}$$

(d)

$$Q = \begin{bmatrix} -2.5 & 2.5 & 0 \\ 2 & -4.5 & 2.5 \\ 0 & 1.5 & -1.5 \end{bmatrix}$$

7. For an irreducible, positive recurrent, time homogeneous continuous-time Markov chain, consider the following statements:

   1 Limiting distribution does not exist.

   2 Stationary distribution exists.

   3 Limiting distribution and stationary distribution both exists and are same.

   Choose the correct option based on the above three statements.

   (a) All three statements are true.
   (b) Statement 1 is always true but 2 and 3 may or may not be true.
   (c) Statement 2 and 3 are always true but 3 is not true.
   (d) None of the statements are necessarily true.

   Answer the following 2 questions based on the problem statement given below.
   Two communication satellites are placed in orbit. The lifetime of the the satellite is exponential distribution with mean $\frac{1}{\mu}$. If one fails its replacement is sent up. The time necessary to prepare and send up a replacement is exponential distribution with mean $\frac{1}{\lambda}$. Let $X(t)$ = the number of satellites not in the orbit at time t. Assume $\{X(t),\ t \geq 0\}$ is a Markov process with state space $\{0, 1, 2\}$.

8. The the infinitesimal generator matrix for $\{X(t),\ t \geq 0\}$ is given by

   (a) $Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$

   (b) $Q = \begin{bmatrix} \lambda & -\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & -2\mu & 2\mu \end{bmatrix}$

   (c) $Q = \begin{bmatrix} -2\mu & 2\mu & 0 \\ \lambda & -(\lambda+\mu) & \mu \\ 0 & \lambda & -\lambda \end{bmatrix}$

(d) $Q = \begin{bmatrix} -\mu & \mu & 0 \\ \lambda & -(\lambda+\mu) & \mu \\ 0 & \lambda & -\lambda \end{bmatrix}$

9. The limiting distribution of the markov chain $\{X(t),\ t \geq 0\}$ is given by

(a) $\pi_0 = \pi_1 = \pi_2 = \frac{1}{4}$

(b) $\pi_0 = \frac{1}{1+2\rho+2\rho^2},\ \pi_1 = 2\rho\pi_0,\ \pi_2 = 2\rho^2\pi_0,$ where $\rho = \frac{\lambda}{\mu}$

(c) $\pi_0 = \frac{1}{1+2\rho+2\rho^2},\ \pi_1 = 2\rho\pi_0,\ \pi_2 = 2\rho^2\pi_0,$ where $\rho = \frac{\mu}{\lambda}$

(d) $\pi_0 = \frac{1}{1+\rho+\frac{\rho^2}{2}},\ \pi_1 = \rho\pi_0,\ \pi_2 = \frac{\rho^2}{2}\pi_0,$ where $\rho = \frac{\mu}{\lambda}$