Stochastic Processes
Assignment (Week 5)

1. Let $X_n$ be the result of $n$th toss of a fair coin such that $X_n = 1$ if head occurs and 0 if tail occurs. Let

$$S_n = \sum_{i=1}^{n} X_i$$

Let $Y_n$ be the remainder when $S_n$ is divided by 2. Which of the following is Not TRUE? (a) $\{Y_n, n = 0, 1, 2, \ldots\}$ is an irreducible Markov chain.
(b) $\{Y_n, n = 0, 1, 2, \ldots\}$ is an aperiodic Markov chain.
(c) $\{Y_n, n = 0, 1, 2, \ldots\}$ has a unique stationary distribution.
(d) limiting distribution of $\{Y_n, n = 0, 1, 2, \ldots\}$ does not exist.

2. Let $X_n$ be the result of $n$th toss of a fair dice. Let $S_n = \sum_{i=1}^{n} X_i$.

Let $Y_n$ be the unit place digit of $S_n$. Which of the following is Not TRUE? (a) $\{Y_n, n = 0, 1, 2, \ldots\}$ is an irreducible Markov chain.
(b) $\{Y_n, n = 0, 1, 2, \ldots\}$ is an aperiodic Markov chain.
(c) $\{Y_n, n = 0, 1, 2, \ldots\}$ has infinitely many stationary distribution.
(d) $\lim_{n \to \infty} P(Y_n = 1) = \frac{1}{6}$.

3. Consider a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $S = \{1, 2, 3, 4\}$ and transition probability matrix $P = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & \frac{1}{3} & 0 & \frac{2}{3} \\
0 & \frac{3}{4} & 0 & \frac{1}{4} \\
\frac{1}{5} & 0 & \frac{7}{5} & 0
\end{pmatrix}$

Which one of the following is a stationary distribution for the Markov chain?
(a) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
(b) $\left(\frac{1}{2}, 0, 0, \frac{3}{4}\right)$
(c) $\left(0, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$
(d) $\left(\frac{1}{5}, 0, \frac{4}{5}, \frac{1}{5}\right)$.

4. Consider a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $\{1, 2\}$. Which one of the following can be TRUE?

(a) The chain has unique stationary distribution.
(b) The chain is always irreducible.
(c) $P(X_n = 1 \mid X_0 = 1)$ does not converges as $n \to \infty$.
(d) Both the states are transient states.

5. Consider a Markov chain $\{X_n, n = 1, 2, \ldots\}$ with state space $S = \{1, 2, 3, 4, 5\}$ and one-step transition probability matrix $P = \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{8} & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}$

Which one of the following is FALSE? (a) $\lim_{n \to \infty} \sum_{i=1}^{n} P_{ii} = 1$. (b) $\sum_{n=1}^{\infty} P_{ii}^n < \infty$.
(c) $\lim_{n \to \infty} P_{13}^n = 0$. (d) Chain has infinitely many stationary distributions.

6. Consider a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $S = \{-100, -99, \ldots, 0, \ldots, 99, 100\}$ with transition probabilities $P_{i,i+1} = P_{i,i-1} = 0.5 \forall -99 \leq i \leq 99$ $P_{100,99} = P_{99,100} = P_{-100,100} = P_{100,-100} = P_{-100,-99} = 0.5$

Which one of the following is TRUE?
(a) The chain is reducible.
(b) Only state 0 is positive recurrent.
(c) \( \lim_{n \to \infty} P_{0,10}^n = \lim_{n \to \infty} P_{0,-10}^n = \frac{1}{201} \).
(d) Stationary distribution does not exist.

7. Consider a Markov chain \( \{X_n, n = 0, 1, 2, \ldots\} \) with state space \( S = \{0, 1\} \) and one-step transition probability matrix \( P = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix} \).

Which one of the following is TRUE?

(a) \( \lim_{n \to \infty} P_{i,0}^n \) does not exist for any choice of \( i \in S \).
(b) \( \lim_{n \to \infty} P_{i,0}^n \) exists and is positive for every choice of \( i \in S \).
(c) \( \lim_{n \to \infty} P_{i,0}^n \) always exists but may be 0 for some choice of \( i \in S \).
(d) \( \lim_{n \to \infty} P_{i,1}^n \) always exists but may be 0 for some choice of \( i \in S \).

8. A total of \( m \) red and \( m \) white balls are distributed into two urns with \( m \) balls per urn. At each step, a ball is randomly selected from each urn and the two balls are interchanged. Let \( X_n \) be number of red balls in urn 1 after \( n \)th step.

Which one of the following is TRUE?

(a) The chain is time reversible and its stationary distribution is \( \pi_i = \frac{m \binom{m}{i} \binom{m}{m-i}}{2m \binom{m}{m}}, i \in \{0, 1, 2, \ldots, m\} \).
(b) The stationary distribution does not exist.
(c) The chain is not time reversible.
(d) The chain is not time reversible but its stationary distribution is \( \pi_i = \frac{m \binom{m}{i} \binom{m}{m-i}}{2m \binom{m}{m}}, i \in \{0, 1, 2, \ldots, m\} \).