Stochastic Processes
Assignment (Week 5)

1. Let X_n be result of nth toss of a fair coin such that $X_n = 1$ if head occurs and 0 if tail occurs. Let

$$S_n = \sum_{i=1}^{n} X_i$$

Let Y_n be the remainder when S_n is divided by 2. Which of the following is Not TRUE? (a) $\{Y_n, n = 0, 1, 2, \ldots\}$ is an irreducible Markov chain.
(b) $\{Y_n, n = 0, 1, 2, \ldots\}$ is an aperiodic Markov chain.
(c) $\{Y_n, n = 0, 1, 2, \ldots\}$ has a unique stationary distribution.
(d) limiting distribution of $\{Y_n, n = 0, 1, 2, \ldots\}$ does not exist.

2. Let X_n be result of nth toss of a fair dice. Let

$$S_n = \sum_{i=1}^{n} X_i$$

Let Y_n be the unit place digit of S_n. Which of the following is Not TRUE? (a) $\{Y_n, n = 0, 1, 2, \ldots\}$ is an irreducible Markov chain.
(b) $\{Y_n, n = 0, 1, 2, \ldots\}$ is an aperiodic Markov chain.
(c) $\{Y_n, n = 0, 1, 2, \ldots\}$ has infinitely many stationary distribution.
(d) $\lim_{n\to\infty} P(Y_n = 1) = \frac{1}{10}$.

3. Consider a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $S = \{1, 2, 3, 4\}$ and transition probability matrix $P = \begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{1}{2} \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & 0 & \frac{7}{8} & 0 \end{pmatrix}$
Which one of the following is a stationary distribution for the Makov chain?
(a) $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ (b) $(\frac{1}{3}, 0, 0, \frac{2}{3})$
(c) $(0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ (d) $(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$.

4. Consider a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $\{1, 2\}$. Which one of the following can be TRUE?
   (a) The chain has unique stationary distribution.
   (b) The chain is always irreducible.
   (c) $P(X_n = 1 \mid X_0 = 1)$ does not converges as $n \to \infty$.
   (d) Both the states are transient states.

5. Consider a Markov chain $\{X_n, n = 1, 2, \ldots\}$ with state space $S = \{1, 2, 3, 4, 5\}$ and one-step transition probability matrix $P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{5} & \frac{3}{5} & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$
Which one of the following is FALSE?
(a)$\lim_{n\to\infty} \sum_{i=1}^{4} P_{ii}^n = 1$. (b)$\sum_{n=1}^{\infty} P_{33}^n < \infty$.
(c) $\lim_{n\to\infty} P_{13}^n = 0$. (d)Chain has infinitely many stationary distributions.

6. Consider a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $S = \{-100, -99, \ldots, 0, \ldots, 99, 100\}$ with transition probabilities $P_{i,i+1} = P_{i,i-1} = 0.5 \ \forall \ -99 \le i \le 99$ $P_{100,99} = P_{100,100} = P_{-100,-100} = P_{-100,-99} = 0.5$
Which one of the following is TRUE?

(a) The chain is reducible.
(b) Only state 0 is positive recurrent.
(c) $\lim_{n\to\infty} P^n_{0,10} = \lim_{n\to\infty} P^n_{0,-10} = \frac{1}{201}$.
(d) Stationary distribution does not exist.

7. Consider a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $S = \{0, 1\}$ and one-step transition probability matrix $P = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix}$

Which one of the following is TRUE?

(a) $\lim_{n\to\infty} P^n_{i,0}$ does not exist for any choice of $i \in S$.
(b) $\lim_{n\to\infty} P^n_{i,0}$ exists and is positive for every choice of $i \in S$.
(c) $\lim_{n\to\infty} P^n_{i,0}$ always exists but may be 0 for some choice of $i \in S$.
(d) $\lim_{n\to\infty} P^n_{i,1}$ always exists but may be 0 for some choice of $i \in S$.

8. A total of m red and m white balls are distributed into two urns with m balls per urn. At each step, a ball is randomly selected from each urn and the two balls are interchanged. Let X_n be number of red balls in urn 1 after n^{th} step.

Which one of the following is TRUE?

(a) The chain is time reversible and its stationary distribution is $\pi_i = \frac{{}^mC_i \, {}^mC_{m-i}}{{}^{2m}C_m}, i \in \{0, 1, 2, \ldots, m\}$.
(b) The stationary distribution does not exist.
(c) The chain is not time reversible.
(d) The chain is not time reversible but its stationary distribution is $\pi_i = \frac{{}^mC_i \, {}^mC_{m-i}}{{}^{2m}C_m}, i \in \{0, 1, 2, \ldots, m\}$.