(1) Find and classify the critical points of the function \( f : \mathbb{R}^2 \to \mathbb{R} \) defined as
\[
 f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4, \quad \text{for all } (x, y) \in \mathbb{R}^2.
\]

Solution. We get
\[
 \frac{\partial f}{\partial x}(x, y) = 20xy - 10x - 4x^3, \quad \frac{\partial f}{\partial y}(x, y) = 10x^2 - 8y - 8y^3.
\]
So then \( \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0 \) implies
\[
 20xy - 10x - 4x^3 = 10x^2 - 8y - 8y^3 = 0.
\]
Solving gives the solution set as
\[
 \{(0, 0), (-0.857, 0.647), (0.857, 0.647), (-2.644, 1.898), (2.644, 1.898)\}.
\]
The discriminant is \( \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \), which is equal to
\[
 (20y - 10 - 12x^2)(-8 - 24y^2) - (20x)^2 = 288x^2y^2 - 304x^2 - 480y^3 + 240y^2 - 160y + 80.
\]
Evaluate the discriminant at each critical point and use second derivative test.
The points of local maxima are \{((0, 0), (-2.644, 1.898), (2.644, 1.898))\}. The saddle points are \{(-0.857, 0.647), (0.857, 0.647)\}. \square

(2) Find the points of local maximum and local minimum and saddle points of the function \( f : \mathbb{R}^2 \to \mathbb{R} \) defined as
\[
 f(x, y) = xy e^{-x^2-y^2}, \quad \text{for all } (x, y) \in \mathbb{R}^2.
\]

Solution. We have
\[
 \frac{\partial f}{\partial x} = e^{-x^2-y^2}(y - 2x^2y), \quad \frac{\partial f}{\partial y} = e^{-x^2-y^2}(x - 2xy^2).
\]
Also the discriminant is
\[ \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 = e^{-2(x^2+y^2)}(-4x^4(2y^2 + 1) + x^2(-8y^4 + 20y^2 + 4) - (1 - 2y^2)^2). \]

So we get the critical points as
\[ \left\{ (0,0), \left( \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}} \right), \left( \pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}} \right) \right\}. \]

Evaluate the discriminant at the critical points.
The points of local maxima are \((\pm 1/\sqrt{2}, \pm 1/\sqrt{2})\). The points of local minima are \((\pm 1/\sqrt{2}, \mp 1/\sqrt{2})\). The saddle point is \((0,0)\). \hfill \square

(3) Find the maximum and minimum values of the function \(f: \mathbb{R}^2 \to \mathbb{R}\) defined as
\[ f(x,y) = xy^2, \quad \text{for all } (x,y) \in \mathbb{R}^2, \]
in the region \(D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y^2 \leq 3\} \).

Solution. We get
\[ \frac{\partial f}{\partial x} = y^2, \quad \frac{\partial f}{\partial y} = 2xy. \]
The critical points are \(\{(x,0) : x \in [0,3]\} \cup \{(0,y) : y \in [0,3]\}\). However, the critical points are all contained in the boundary of \(D\), which is
\[ \partial D = \{(x,0) : x \in [0,3]\} \cup \{(0,y) : y \in [0,3]\} \cup \{(x,y) : x \geq 0, y \geq 0, x^2 + y^2 = 3\} \]
We have \(f(x,0) = f(y,0) = 0\), for all \(x, y \in [0,3]\). Further, for any \((x,y)\) such that \(x \geq 0, y \geq 0, x^2 + y^2 = 3\), we have
\[ f(x,y) = xy^2 = x(3-x^2) = 3x-x^3, \]
which is maximized at \((1,\sqrt{2})\), with the maximum value 2. So the minimum value of \(f\) is 0 and the maximum value of \(f\) is 2. \hfill \square

(4) Using Lagrange multipliers, find the maximum value of the function \(f(x,y,z) = x^2 + y^2 + z^2\) subject to the constraints \(x + y + z = 12\).

Solution. Let \(g(x,y,z) = x + y + z - 12\). So the Lagrangian function is
\[ L(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z) = x^2 + y^2 + z^2 - \lambda x - \lambda y - \lambda z + 12\lambda. \]

We then have
\[ \frac{\partial L}{\partial x} = 2x - \lambda, \quad \frac{\partial L}{\partial y} = 2y - \lambda, \quad \frac{\partial L}{\partial z} = 2z - \lambda, \quad \frac{\partial L}{\partial \lambda} = -x - y - z + 12. \]

So the stationary point \((x_0,y_0,z_0,\lambda_0)\) is given by
\[ \frac{\partial L}{\partial x}(x_0,y_0,z_0,\lambda_0) = \frac{\partial L}{\partial y}(x_0,y_0,z_0,\lambda_0) = \frac{\partial L}{\partial z}(x_0,y_0,z_0,\lambda_0) = \frac{\partial L}{\partial \lambda}(x_0,y_0,z_0,\lambda_0) = 0, \]
which gives \(x_0 = y_0 = z_0 = \lambda_0/2 = 4\). It then follows that \(f\) attains its maximum at \((4,4,4)\) and the maximum value is 48. \hfill \square
(5) (a) For any \( k \in \mathbb{R} \) and \( R = [a, b] \times [c, d] \), show that

\[
\iint_{R} k \, dA = k(b - a)(d - c). \tag{1}
\]

**Solution.** By Fubini Theorem, we have

\[
\iint_{R} k \, dA = \int_{a}^{b} \int_{c}^{d} k \, dy \, dx = \int_{a}^{b} k(d - c) \, dx = k(b - a)(d - c). \tag{2}
\]

(b) Use (a) to show that

\[0 \leq \iint_{R} \sin \pi x \cos \pi y \, dA \leq \frac{1}{32},\]

where \( R = [0, 1/4] \times [1/4, 1/2] \). \tag{3}

**Solution.** We first note that if \((x, y) \in [0, 1/4] \times [1/4, 1/2]\), then \( \sin \pi x \cos \pi y \geq 0 \). So \( \iint_{R} \sin \pi x \cos \pi y \, dA \geq 0 \). Now let \( f(x, y) = \sin \pi x \cos \pi y \), \((x, y) \in [0, 1/4] \times [1/4, 1/2]\). We have

\[
\frac{\partial f}{\partial x} = \pi \cos \pi x \cos \pi y, \quad \frac{\partial f}{\partial y} = -\pi \sin \pi x \sin \pi y.
\]

Then there are no critical points of \( f \) in the region \([0, 1/4] \times [1/4, 1/2]\). So the maximum of \( f \) on \([0, 1/4] \times [1/4, 1/2]\) is attained in the boundary set

\[
\{0, 1/4\} \times [1/4, 1/2] \cup [0, 1/4] \times \{1/4, 1/2\}.
\]

Now \( \sin \pi x \) is increasing for \( x \in [0, 1/4] \) and \( \cos \pi y \) is decreasing for \( y \in [1/4, 1/2] \). So \( \sin \pi x \) has its maximum at \( x = 1/4 \) and \( \cos \pi y \) has its maximum at \( y = 1/4 \). So \( f \) has its maximum at the point \((1/4, 1/4)\) and the maximum value is \( f(1/4, 1/4) = \sin(\pi/4) \cos(\pi/4) = 1/2 \).

Then, using (a), we get

\[
\iint_{R} \sin \pi x \cos \pi y \, dA \leq \iint_{R} \frac{1}{2} \, dA = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{32}. \tag{4}
\]