(1) A subset $A \subseteq \mathbb{R}$ is defined to be convex if for any $x, y \in A$,
\[(1 - t)x + ty \in A, \quad \text{for all } t \in [0, 1].\]

Show that $A \subseteq \mathbb{R}$ is convex if and only if $A$ is an interval. \[2\]

**Solution.** Consider any $a, b \in A$, $a \leq b$ and $c \in \mathbb{R}$ such that $a \leq c \leq b$. Thus $c \in [a, b]$. Then the equation $(1 - t)a + tb = c$, $t \in [0, 1]$ has the solution $t = \frac{c-a}{b-a} \in [0, 1]$. Then it follows that $A$ is an interval if and only if $A$ is convex. \[\Box\]

(2) Prove or disprove. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $U \subseteq \mathbb{R}$ be open. Then $f(U)$ is open. \[2\]

**Solution.** False. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = 0$, for all $x \in \mathbb{R}$. Then for any open $U \subseteq \mathbb{R}$, we have $f(U) = \{0\}$, which is not open. \[\Box\]

(3) Let $K \subseteq \mathbb{R}$ be nonempty compact and let $f : K \to K$ be continuous such that
\[|f(x) - f(y)| \geq |x - y|, \quad \text{for all } x, y \in K.\]

Prove that
(a) $f$ is one-one and $f^{-1} : f(A) \to A$ is continuous. \[2\]
(b) $f(A) = A$. \[2\]

**Correct Question.** Let $K \subseteq \mathbb{R}$ be nonempty compact and let $f : K \to K$ be continuous such that
\[|f(x) - f(y)| \geq |x - y|, \quad \text{for all } x, y \in K.\]

Prove that
(a) $f$ is one-one and $f^{-1} : f(K) \to K$ is continuous. \[2\]
(b) $f(K) = K$. \[2\]

**Solution.**
(a) Since $|f(x) - f(y)| \geq |x - y|$, for all $x, y \in K$, if $x, y \in K$, $x \neq y$, then $|x - y| > 0$ and so $|f(x) - f(y)| > 0$, which implies $f(x) \neq f(y)$. Thus $f$ is one-one. Further, we get

$$|f^{-1}(x) - f^{-1}(y)| \leq |f(f^{-1}(x)) - f(f^{-1}(y))| = |x - y|, \quad \text{for all } x, y \in f(K).$$

So for any $x_0 \in f(K)$ and sequence $(x_n)$ in $f(K)$ such that $x_n \to x_0$, we have $|x_n - x_0| \to 0$, and so $|f^{-1}(x_n) - f^{-1}(x_0)| \to 0$. Thus $f^{-1}(x_n) \to f^{-1}(x_0)$. Hence $f^{-1} : f(K) \to K$ is continuous. \hfill \Box

(b) Clearly $f(K) \subseteq K$. Since $K$ is compact, $K$ is closed and bounded. Let $\alpha = \inf K$ and $\beta = \sup K$. Then $\alpha, \beta \in K$. Note that $|f(\alpha) - f(\beta)| \geq |\alpha - \beta|$. So $|f(\alpha) - f(\beta)| = |\alpha - \beta|$. Then $\{f(\alpha), f(\beta)\} = \{\alpha, \beta\}$. Without loss of generality, suppose $f(\alpha) = \alpha$, $f(\beta) = \beta$. Consider any $\gamma \in K$. Then $\gamma \in [\alpha, \beta]$. So we have

$$|f(\alpha) - f(\gamma)| \geq |\alpha - \gamma| \quad \text{and} \quad |f(\gamma) - f(\beta)| \geq |\gamma - \beta|,$$

which forces the equalities,

$$|f(\alpha) - f(\gamma)| = |\alpha - \gamma| \quad \text{and} \quad |f(\gamma) - f(\beta)| = |\gamma - \beta|.$$

This forces $f(\gamma) = \gamma$. Thus $\gamma \in f(K)$. So $K \subseteq f(K)$. Hence $f(K) = K$. \hfill \Box

(4) Prove that $Q$ is disconnected in $\mathbb{R}$. \hfill [2]

**Solution.** We know that $\sqrt{2} \notin Q$. Let $A = (-\infty, \sqrt{2})$ and $B = (\sqrt{2}, \infty)$. So $A, B$ are open, nonempty, disjoint and $(A \cup B) \cap Q = Q$. So $Q$ is disconnected. \hfill \Box

(5) Let $B(\mathbb{R}) = \{A \subseteq \mathbb{R} : A \text{ is bounded}\}$. For any nonempty bounded set $A \in B(\mathbb{R})$, define the **diameter** of $A$ as

$$\text{diam}(A) = \sup\{|x - y| : x, y \in A\}.$$

Further, define $\text{diam}(\emptyset) = 0$. For any $A \in B(\mathbb{R})$, define

$$\alpha(A) = \inf \left\{ r > 0 : A \subseteq \bigcup_{i=1}^{n} A_i, \text{ for } A_i \in B(\mathbb{R}) \text{ with } \text{diam}(A_i) \leq r, \forall i \in \{1, \ldots, n\} \right\}.$$

Show that

(a) If $K \subseteq \mathbb{R}$ is compact, then $\alpha(K) = 0$. \hfill [3]

**Solution.** If $K \subseteq \mathbb{R}$ is compact, then $K \in B(\mathbb{R})$, since $K$ is closed and bounded. Let $a = \inf K$, $b = \sup K$. Then $K \subseteq [a, b]$. We prove that $\alpha([a, b]) = 0$. For any $n \in \mathbb{Z}^+$ we have

$$[a, b] = \left[ a, a + \frac{b-a}{n} \right] \cup \left[ a + \frac{b-a}{n}, a + \frac{2(b-a)}{n} \right] \cup \cdots \cup \left[ a + \frac{(n-1)(b-a)}{n}, b \right].$$

We then conclude that $\alpha([a, b]) = 0$. Then using (b), we get $\alpha(K) = 0$. \hfill \Box

(b) If $A, B \in B(\mathbb{R})$, $A \subseteq B$, then $\alpha(A) \leq \alpha(B)$. \hfill [2]

**Solution.** Let $A, B \in B(\mathbb{R})$, $A \subseteq B$. Let $r > 0$ and $B_1, \ldots, B_n$ such that $\text{diam}(B_i) \leq r$ and $B \subseteq \bigcup_{i=1}^{n} B_i$. Then $A \subseteq \bigcup_{i=1}^{n} B_i$. It then follows that $\alpha(A) \subseteq \alpha(B)$. \hfill \Box
(6) Show that if \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and periodic, then \( f \) is uniformly continuous. \[ \square \]

**Solution.** Let \( p > 0 \) such that \( f(x + p) = f(x) \), for all \( x \in \mathbb{R} \). Consider any \( \epsilon > 0 \). Since \( f \) is continuous, \( f \) is uniformly continuous on \([0, p]\). So there exists \( \delta > 0 \) such that if \( x, y \in [0, p] \) and \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \). We now write

\[
R = \bigsqcup_{n \in \mathbb{Z}} [np, (n + 1)p].
\]

Let \( \delta' = \min\{\delta, p/2\} > 0 \). Now consider any \( x, y \in \mathbb{R} \), \( x < y \) such that \( |x - y| < \delta' \). Then either \( x, y \in [np, (n + 1)p] \) for some \( n \in \mathbb{Z} \), or \( x \in [np, (n + 1)p], y \in [(n + 1)p, (n + 2)p] \).

Suppose \( x, y \in [np, (n + 1)p] \), for some \( n \in \mathbb{Z} \). Then define \( x' = x - np, y' = y - np \). So \( x', y' \in [0, p] \) and \( |x' - y'| = |x - y| < \delta' \). So \( |f(x') - f(y')| < \epsilon \). Then by periodicity, we get \( |f(x) - f(y)| = |f(x') - f(y')| < \epsilon \).

Now suppose \( x \in [np, (n + 1)p], y \in [(n + 1)p, (n + 2)p] \), for some \( n \in \mathbb{Z} \). Then define \( x' = x - np, y' = y - (n + 1)p \). So \( x', y' \in [0, p] \) and \( \max\{|x' - y'|, |x - y|\} < \delta' \), by choice of \( \delta' \). So \( |f(x') - f(y')| < \epsilon \). Then by periodicity, \( |f(x) - f(y)| = |f(x') - f(y')| < \epsilon \). \( \square \)

(7) Let \( f : (a, b) \rightarrow \mathbb{R} \) be uniformly continuous. Then show that there exists \( \tilde{f} : [a, b] \rightarrow \mathbb{R} \) such that \( \tilde{f} \) is uniformly continuous and \( \tilde{f}(x) = f(x) \), for all \( x \in (a, b) \).

**Solution.** It is enough to prove that the limits \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to b^-} f(x) \) both exist.

Consider any sequence \((x_n)\) in \((a, b)\) such that \( x_n \to a \). Consider any \( \epsilon > 0 \). Since \( f \) is uniformly continuous on \((a, b)\), there exists \( \delta > 0 \) such that if \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \). Since \( x_n \to a \), \((x_n)\) is Cauchy. So there exists \( n_0 \in \mathbb{Z}^+ \) such that \( |x_n - x_m| < \delta \), for all \( n, m \geq n_0 \). Then we get \( |f(x_n) - f(x_m)| < \epsilon \), for all \( n, m \geq n_0 \). This proves that \((f(x_n))\) is Cauchy. Hence \( f(x_n) \) is convergent. Let \( \lambda = \lim_{n \to \infty} f(x_n) \).

Now consider any sequence \((y_n)\) in \((a, b)\) such that \( y_n \to a \), \((f(y_n))\) is convergent and let \( \lambda' = \lim_{n \to \infty} f(y_n) \). Define \( z_n = x_n \) if \( n \) is odd, and \( z_n = y_n \) if \( n \) is even. Then it follows that \( \lambda = \lambda' \). Thus \( \lim_{x \to a^+} f(x) \) exists.

Similarly, we can argue that \( \lim_{x \to b^-} f(x) \) exists. \( \square \)

(8) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = x^2 \), for all \( x \in \mathbb{R} \). Show that \( f \) is uniformly continuous on \([0, 1]\) and \( f \) is not uniformly continuous on \( \mathbb{R} \).

**Solution.** Since \( f \) is continuous on \([0, 1]\), \( f \) is uniformly continuous on \([0, 1]\). For any \( x \in [0, \infty) \) and \( n \in \mathbb{Z}^+ \), we have

\[
|f(x) - f\left(x + \frac{1}{n}\right)| = f\left(x + \frac{1}{n}\right) - f(x) = \left(x + \frac{1}{n}\right)^2 - x^2 = \frac{2x}{n} + \frac{1}{n^2}.
\]

Further, we have

\[
\frac{2x}{n} + \frac{1}{n^2} \geq 1 \iff 2xn + 1 \geq n^2 \iff x \geq \frac{n^2 - 1}{2n}.
\]

Therefore, \( f \) is not uniformly continuous on \( \mathbb{R} \).
Now consider any $\delta > 0$. Then there exists $n_0 \in \mathbb{Z}^+$ such that $\frac{1}{n_0} < \delta$. Choose $x_0 = \frac{n_0^2 - 1}{2n_0}$. Then we have

$$\left| x_0 - \left( x_0 + \frac{1}{n_0} \right) \right| < \delta \quad \text{and} \quad \left| f(x_0) - f\left( x_0 + \frac{1}{n_0} \right) \right| = \frac{2x_0}{n_0} + \frac{1}{n_0^2} \geq 1.$$

Thus $f$ is not uniformly continuous on $\mathbb{R}$. \qed