1. A subset $A \subseteq \mathbb{R}$ is defined to be convex if for any $x, y \in A$,

$$(1 - t)x + ty \in A, \quad \text{for all } t \in [0, 1].$$

Show that $A \subseteq \mathbb{R}$ is convex if and only if $A$ is an interval. [2]

2. Prove or disprove. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $U \subseteq \mathbb{R}$ be open. Then $f(U)$ is open. [2]

3. Let $K \subseteq \mathbb{R}$ be nonempty compact and let $f : K \rightarrow K$ be continuous such that

$$|f(x) - f(y)| \geq |x - y|, \quad \text{for all } x, y \in K.$$

Prove that

(a) $f$ is one-one and $f^{-1} : f(A) \rightarrow A$ is continuous. [2]

(b) $f(A) = A$. [2]

4. Prove that $\mathbb{Q}$ is disconnected in $\mathbb{R}$. [2]

5. Let $B(\mathbb{R}) = \{A \subseteq \mathbb{R} : A$ is bounded\}. For any nonempty bounded set $A \in B(\mathbb{R})$, define the diameter of $A$ as

$$\text{diam}(A) = \sup\{|x - y| : x, y \in A\}.$$

Further, define $\text{diam}(\emptyset) = 0$. For any $A \in B(\mathbb{R})$, define

$$\alpha(A) = \inf\left\{r > 0 : D \subseteq \bigcup_{i=1}^{n} A_i, \ \text{for } A_i \in B(\mathbb{R}) \ \text{with} \ \text{diam}(A_i) \leq r, \ \forall i \in \{1, \ldots, n\}\right\}.$$

Show that

(a) If $K \subseteq$ is compact, then $\alpha(K) = 0$. [3]

(b) If $A, B \in B(\mathbb{R}), A \subseteq B$, then $\alpha(A) \leq \alpha(B)$. [2]

(c) If $A \in B(\mathbb{R})$, then $\alpha(A) = \alpha(A)$. [3]
(6) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, then $f$ is uniformly continuous. [3]

(7) Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous. Then show that there exists $\tilde{f} : [a, b] \to \mathbb{R}$ such that $\tilde{f}$ is uniformly continuous and $\tilde{f}(x) = f(x)$, for all $x \in (a, b)$. [4]

(8) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, for all $x \in \mathbb{R}$. Show that $f$ is uniformly continuous on $[0, 1]$ and $f$ is not uniformly continuous on $\mathbb{R}$. [5]