Let $A \subseteq \mathbb{R}$ be nonempty bounded above. Define

$$-A = \{-a : a \in A\}.$$ 

Show that $-A$ is bounded below and $\inf(-A) = -\sup(A)$. \[2\]

**Solution.** Let $M \in \mathbb{R}$ such that $a \leq M$, for all $a \in A$. So $-M \leq -a$, for all $a \in A$, that is, $-M \leq b$, for all $b \in -A$. So $-A$ is bounded below. Let $\alpha = \sup(A)$. Then we have $\alpha \leq M$, which gives $-M \leq -\alpha$. Further, if there exists $b \in -A$ such that $-M \leq b < -\alpha$, then $\alpha < -b \leq M$ and $-b \in A$, which is not possible. Thus $-\alpha = \inf(-A)$. \[\Box\]

(2) Let $I_n = [a_n, b_n]$, $n \in \mathbb{Z}^+$ be a sequence of closed and bounded intervals in $\mathbb{R}$ such that $I_n \cap I_m \neq \emptyset$, for any $m, n \in \mathbb{Z}^+$. Show that $\bigcap_{n \in \mathbb{Z}^+} I_n$ is a nonempty closed interval in $\mathbb{R}$. \[3\]

**Solution.** Let $J_n = \bigcap_{j=1}^n I_j$, for all $n \in \mathbb{Z}^+$. Then by the given condition, we have $J_{n+1} \subseteq J_n$, for all $n \in \mathbb{Z}^+$. Also $\bigcap_{n \in \mathbb{Z}^+} I_n = \bigcap_{n \in \mathbb{Z}^+} J_n$.

We now show that each $J_n$ is a closed and bounded interval. Clearly $J_1 = I_1$ is a closed and bounded interval. Suppose $J_n$ is a closed and bounded interval, for some $n \in \mathbb{Z}^+$. Then we have $J_{n+1} = J_n \cap [a_{n+1}, b_{n+1}]$, which means $J_{n+1}$ is closed and bounded. Now consider any $u, v \in J_{n+1}$, $u \leq v$ and any $w \in \mathbb{R}$ such that $u \leq w \leq v$. Since we have $u, v \in [a_{n+1}, b_{n+1}]$ and $[a_{n+1}, b_{n+1}]$ is an interval, we get $w \in [a_{n+1}, b_{n+1}]$. By induction, since $J_n$ is an interval and we have $u, v \in J_n$, we get $w \in J_n$. Thus $w \in J_{n+1}$. Hence $J_{n+1}$ is an interval.

Let $J_n = [\alpha_n, \beta_n]$, $n \in \mathbb{Z}^+$. Then since $J_{n+1} \subseteq J_n$, for all $n \in \mathbb{Z}^+$, we conclude that the sequence $(\alpha_n)$ is increasing and the sequence $(\beta_n)$ is decreasing. Further, $\alpha_n \leq \beta_1$ and $\alpha_1 \leq \beta_n$, for all $n \in \mathbb{Z}^+$. So the sequence $(\alpha_n)$ is bounded above and the sequence $(\beta_n)$ is bounded below. Thus $(\alpha_n)$ and $(\beta_n)$ are convergent.

Let $\alpha_n \to \alpha$ and $\beta_n \to \beta$. It follows that $\alpha, \beta \in J_n$, for all $n \in \mathbb{Z}^+$ and $\alpha \leq \beta$. Thus $[\alpha, \beta] \subseteq \bigcap_{n \in \mathbb{Z}^+} J_n$ and so $\bigcap_{j \in \mathbb{Z}^+} J_n$ is nonempty. Since each $J_n$ is closed and bounded, $\bigcap_{j \in \mathbb{Z}^+} J_n$ is also closed and bounded. It follows that $\bigcap_{j \in \mathbb{Z}^+} J_n = [\alpha, \beta]$, and so the intersection is a closed and bounded interval. \[\Box\]
(3) Let $\mathbb{R} \setminus \mathbb{Q}$. Is $A$ closed? Justify.

**Correct Question.** Let $A = \mathbb{R} \setminus \mathbb{Q}$. Is $A$ closed? Justify.

**Solution.** We have $\mathbb{R} \setminus A = \mathbb{Q}$. Consider any $a \in \mathbb{Q}$ and $r > 0$. Since $\frac{\sqrt{2}}{n} \to 0$, there exists $n_0 \in \mathbb{Z}^+$ such that $a + \frac{\sqrt{2}}{n_0} \in (a - r, a + r)$. But since $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, $a + \frac{\sqrt{2}}{n_0} \not\in \mathbb{R} \setminus \mathbb{Q}$. Thus $(a - r, a + r \not\subseteq \mathbb{Q})$, for any $r > 0$. Thus $\mathbb{Q}$ is not open. So $A = \mathbb{R} \setminus \mathbb{Q}$ is not closed.

(4) Let $A, B \subseteq \mathbb{R}$.

(a) If $A$ is open and $B$ is closed, show that $A \setminus B$ is open and $B \setminus A$ is closed.

**Solution.** Since $B$ is closed, $\mathbb{R} \setminus B$ is open. Then $A \setminus B = A \cap (\mathbb{R} \setminus B)$ is open.

Since $A$ is open, $\mathbb{R} \setminus A$ is closed. Then $B \setminus A = B \cap (\mathbb{R} \setminus A)$ is closed.

(b) Define $AB = \{xy : x \in A, y \in B\}$. If $A$ is open and $B$ is arbitrary, must $AB$ be open? Justify.

**Solution.** No. Let $A = (-1, 1)$ and $B = \{0\}$. Then $A$ is open and $AB = \{0\}$ is not open.

(5) Which of the following sets are compact? Justify.

(a) $[0, 1] \cup [5, 6]$.

**Solution.** Since $[0, 1]$ and $[5, 6]$ are closed and bounded, $[0, 1] \cup [5, 6]$ is closed and bounded. So $[0, 1] \cup [5, 6]$ is compact.

(b) $\{x \in \mathbb{R} : x \geq 0\}$.

**Solution.** Let $A = \{x \in \mathbb{R} : x \geq 0\} = [0, \infty)$. Then $\mathbb{R} \setminus A = (-\infty, 0)$ is open. So $A$ is closed. But $A$ is not bounded. Thus $A$ is not compact.

(c) $\{x \in [0, 1] : x \in \mathbb{R} \setminus \mathbb{Q}\}$.

**Solution.** Let $A = \{x \in [0, 1] : x \in \mathbb{R} \setminus \mathbb{Q}\}$. Then clearly $A$ is bounded. But, as in (3), $A$ is not closed. Thus $A$ is not compact.

(d) $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{Z}^+ \right\}$.

**Solution.** Let $A = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{Z}^+ \right\}$. Then clearly $A \subseteq [0, 1]$ and so $A$ is bounded. Now we get

$$\mathbb{R} \setminus A = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{n \in \mathbb{Z}^+} \left(\frac{1}{n+1}, \frac{1}{n}\right).$$

So $\mathbb{R} \setminus A$ is open, that is, $A$ is closed. Thus $A$ is compact.