(1) Show that $\sqrt{p}$ is irrational, for any prime $p$. 

**Solution.** Consider any prime $p$. Suppose $\sqrt{p}$ is rational. Let $\sqrt{p} = \frac{r}{s}$, where $r, s \in \mathbb{Z}^+$ and $r, s$ have no common factors. Then $p = \frac{r^2}{s^2}$, which gives $r^2 = ps^2$. So $p$ divides $r^2$, which means $p$ divides $r$. Let $r = pt$, for some $t \in \mathbb{Z}^+$. Then $r^2 = p^2t^2$. This gives $p^2t^2 = ps^2$, which implies $s^2 = pt^2$. Thus $p$ divides $s^2$ and so $p$ divides $s$. Hence $p$ is a common factor of $r$ and $s$, which is a contradiction. So $\sqrt{p}$ is irrational. 

(2) For the following subsets, (i) check if supremum and infimum exist, (ii) if they exist, find them, (iii) if they exist, check if they belong to the subset.

(a) $S = \{(-1/n) + (1 + (-1)^n)n^2 : n \in \mathbb{Z}^+\}$. 

(b) $S = \{x \in \mathbb{R} : x^2 < 9\}$.

**Solution.**

(a) Let $x_n = (-1/n) + (1 + (-1)^n)n^2$, $n \in \mathbb{Z}^+$. So we have

$$x_n = \begin{cases} 
-\frac{1}{n} + 2n^2 & n \text{ is even} \\
-\frac{1}{n} & n \text{ is odd}
\end{cases}$$

So $-1 \leq x_n$, for all $n \in \mathbb{Z}^+$ and $x_1 = -1$. Thus inf $S = -1$. Further, since $-1/n \rightarrow 0$, the set $\{-1/n + 2n^2 : n \in \mathbb{Z}^+, n \text{ even}\}$ is unbounded above. Thus $S$ is unbounded above. So sup $S$ does not exist.

(b) For any $x \in \mathbb{R}$ such that $|x| \geq 4$, we have $x^2 \geq 16 > 9$. Thus $S \subseteq (-4, 4)$, that is, $S$ is bounded. By definition, we then have sup $S = \sqrt{3}$. Then we note that $x \in S$ if and only if $-x \in S$. So we get inf $S = -\sqrt{3}$.

(3) Let $S, T \subseteq \mathbb{R}$ be nonempty subsets such that $S \cup T = \mathbb{R}$ and $s < t$, for all $s \in S, t \in T$. Show that there exists a unique $\beta \in \mathbb{R}$ such that $s \in S$ for every $s < \beta$, and $t \in T$ for every $t > \beta$. 


Solution. Since \( s < t \), for every \( s \in S, t \in T \), we conclude that \( S \) is bounded above and \( T \) is bounded below. So \( \sup S, \inf T \) both exist. Further, it follows that \( \sup S \leq \inf T \).
Let \( \alpha = \sup S, \gamma = \inf T \). If \( \alpha < \gamma \), then there exists \( u \in \mathbb{R} \) such that \( \alpha < u < \gamma \). This means \( s < u \), for all \( s \in S \) and \( u < t \), for all \( t \in T \). Thus \( u \not\in S \cup T = \mathbb{R} \), a contradiction. So \( \alpha = \gamma \). We take \( \beta = \alpha = \gamma \). Then it follows that \( s \in S \) for every \( s < \beta \), and \( t \in T \) for every \( t > \beta \).

(4) (a) Give an example, with justification, of a union of infinitely many distinct closed subsets of \( \mathbb{R} \) being closed. [2]

Solution. Let \( I_n = [-n, n] \), for every \( n \in \mathbb{Z}^+ \). Then \( I_n \) is a closed interval and hence is a closed set, for every \( n \in \mathbb{Z}^+ \). Further, \( \bigcup_{n \in \mathbb{Z}^+} I_n = \mathbb{R} \), which is a closed set. □

(b) Give an example, with justification, of a union of infinitely many distinct closed subsets of \( \mathbb{R} \) not being closed. [2]

Solution. Let \( I_n = [-1 + \frac{1}{2n+1}, 1 - \frac{1}{2n+1}] \), for every \( n \in \mathbb{Z}^+ \). Then \( I_n \) is a closed interval and hence is a closed set, for every \( n \in \mathbb{Z}^+ \). Further, \( \bigcup_{n \in \mathbb{Z}^+} I_n = (-1, 1) \), which is not a closed set. □

(5) Show that if \( A \subseteq \mathbb{R}, A \neq \emptyset \) is such that \( A \) and \( \mathbb{R} \setminus A \) are both closed, then \( A = \mathbb{R} \). [3]

Solution. if \( A \subseteq \mathbb{R}, A \neq \emptyset \) is such that \( A \) and \( \mathbb{R} \setminus A \) are both closed, then \( A \) and \( \mathbb{R} \setminus A \) are both open. Pick a point \( a \in A \) and a point \( b \in \mathbb{R} \setminus A \). Without loss of generality, assume that \( a < b \). Let \( S = \{ x \in [a, b] : [a, x] \subseteq A \} \). So \( S \) is bounded and let \( u = \sup S \).

Clearly \( u \leq b \).

Now let \( \epsilon > 0 \), and consider the interval \( J = (u-\epsilon, u+\epsilon) \). Since \( u = \sup S \), \( S \cap (u-\epsilon, u] \neq \emptyset \), and therefore \( J \cap S \neq \emptyset \). This shows that \( u \not\in \mathbb{R} \setminus A \). Thus \( u \in S \), and therefore \( u < b \) (since \( b \not\in S \)). Let \( v = \min\{u + \epsilon, b\} \). Clearly \( v \not\in S \), so there is some \( x \in [a, v] \setminus A \). since \( [a, u] \subseteq A \), we must have \( x \in [u, v] \subseteq [u, u+\epsilon] \). Thus, \( J \setminus A \neq \emptyset \), and \( J \not\subseteq A \). Thus \( u \not\in A \). This is a contradiction, showing that \( A \) and \( \mathbb{R} \setminus A \) can’t both be open. □