NPTEL Measure Theory, July 2018
Assignment 9

Deadline: Wednesday, October 3, 2018, 23:59 IST

Q.1. **[6 marks]**

(a) Let $f : [0,1] \to [0,\infty)$ be Riemann integrable on $[\epsilon, 1]$, for all $\epsilon > 0$. Show that

$$f \in L_1[0,1] \iff \lim_{\epsilon \to 0} \int_\epsilon^1 f(x)\,dx \text{ exists.}$$

Further, in that case, show that

$$\int_{[0,1]} f(x)\,d\lambda(x) = \lim_{\epsilon \to 0} \int_\epsilon^1 f(x)\,dx.$$

(b) *(Mean Value Property)* Let $f : [a,b] \to \mathbb{R}$ be a continuous function and let $E \subseteq [a,b]$, $E \in \mathcal{L}$ be such that $\lambda(E) > 0$. Show that there exists $\alpha \in \mathbb{R}$ such that

$$\int_E f(x)\,d\lambda(x) = \alpha\lambda(E).$$

(c) Let $f \in L_1(\mathbb{R})$, $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\alpha < \beta$ in $\mathbb{R}$ such that $\alpha \le g(x) \le \beta$, for a.e. $x(\lambda)$. Show that $fg \in L_1(\mathbb{R})$ and that there exists $\gamma \in [\alpha, \beta]$ such that

$$\int_{\mathbb{R}} |f| g\,d\lambda = \gamma \int_{\mathbb{R}} |f|\,d\lambda.$$

(d) Let $f \in L_1(\mathbb{R})$ and $a \in \mathbb{R}$. Define

$$F(x) = \begin{cases} \int_{[a,x]} f(t)\,d\lambda(t), & x \ge a \\ \int_{[x,a]} f(t)\,d\lambda(t), & x \le a \end{cases}$$

Show that F is continuous.

(e) Let $f : [a, b] \to \mathbb{R}$ be a bounded measurable function. Show that $f \in L_1[a, b]$.

(f) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that $f \in L_1[a, b]$.

Q.2. **[4 marks]**

(a) Give examples to show that the analogues of the monotone convergence theorem and the dominated convergence theorem do not hold for the Riemann integral.

(b) Let $f \in L_1(\mathbb{R})$. Define $g : [0, \infty) \to \mathbb{R}$ as

$$g(t) = \sup \left\{ \int_{\mathbb{R}} \left| f(x + y) - f(x) \right| d\mu(x) : -t \leq y \leq t \right\}.$$

Show that g is continuous at $t = 0$.

Hint. Use the simple function technique.