NPTEL Measure Theory, July 2018
Assignment 8

Deadline: Wednesday, September 26, 2018, 23:59 IST

Q.1. **[4 marks]**

Let $(X, \mathcal{S}, \mu)$ be a measure space.

(a) Let $f : X \to \mathbb{R}$ be nonnegative measurable. If $\int_X f \, d\mu = 0$, then show that $f(x) = 0$, for a.e. $x(\mu)$.

(b) Let $f : X \to \mathbb{R}$ be nonnegative measurable. If $\int_X f \, d\mu < \infty$, then show that $f(x) < \infty$, for a.e. $x(\mu)$.

(c) Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative integrable functions and let f be nonnegative integrable. Show that as $n \to \infty$,

$$\int_X |f_n| \, d\mu \to \int_X |f| \, d\mu \iff \int_X |f_n - f| \, d\mu \to 0.$$

Q.2. **[4 marks]**

Let $(X, \mathcal{S}, \mu)$ be a measure space.

(a) *(Chebyshev Inequality)* For every $\epsilon > 0$ and $f \in L_1(\mu)$, show that

$$\mu\big(\{x \in X : |f(x)| \geq \epsilon\}\big) \leq \frac{1}{\epsilon} \int_X |f| \, d\mu < \infty.$$

(b) Let $f \in L_1(\mu)$ and let $M > 0$ such that

$$\left| \frac{1}{\mu(E)} \int_E f \, d\mu \right| \leq M, \quad \text{for every } E \in \mathcal{S} \text{ with } 0 < \mu(E) < \infty.$$

Show that $|f(x)| \leq M$, for a.e. $x(\mu)$.

Q.3. **[2 marks]**

Let $(X, \mathcal{S})$ be a measurable space and $f : X \to \mathbb{R}$ be $\mathcal{S}$-measurable. Let $\mathcal{S}_0 = \{f^{-1}(E) : E \in \mathcal{B}_{\mathbb{R}}\}$. We know that $\mathcal{S}_0 \subseteq \mathcal{S}$ and $\mathcal{S}_0$ is a σ-algebra on X. Prove the following.

(a) If $\phi : \mathbb{R} \to \mathbb{R}$ is Borel measurable, that is, $\phi^{-1}(E) \in \mathcal{B}_{\mathbb{R}}$, for every $E \in \mathcal{B}_{\mathbb{R}}$, then $\phi \circ f : X \to \mathbb{R}$ is $\mathcal{S}_0$-measurable.

(b) If $\psi : X \to \mathbb{R}$ is $\mathcal{S}_0$-measurable, then there exists a Borel measurable function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\psi = \phi \circ f$.

Hint. Use the simple function technique and note that if ψ is a simple $\mathcal{S}_0$-measurable function, then $\psi = \sum_{i=1}^{n} a_n \chi_{f^{-1}(E_i)}$, for some $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ and $E_i \in \mathcal{R}_{\mathbb{R}}$, for $i = 1, \ldots, n$. Then show that $\psi = \left(\sum_{i=1}^{n} a_i \chi_{E_i}\right) \circ f$.