Q.1. [4 marks]
Let \((X, S, \mu)\) be a measure space.

(a) Let \(f : X \to \mathbb{R}\) be nonnegative measurable. If \(\int_X f \, d\mu = 0\), then show that \(f(x) = 0\), for a.e. \(x(\mu)\).

(b) Let \(f : X \to \mathbb{R}\) be nonnegative measurable. If \(\int_X f \, d\mu < \infty\), then show that \(f(x) < \infty\), for a.e. \(x(\mu)\).

(c) Let \((f_n)_{n=1}^{\infty}\) be a sequence of nonnegative integrable functions and let \(f\) be nonnegative integrable. Show that as \(n \to \infty\),
\[
\int_X |f_n| \, d\mu \to \int_X |f| \, d\mu \iff \int_X |f_n - f| \, d\mu \to 0.
\]

Q.2. [4 marks]
Let \((X, S, \mu)\) be a measure space.

(a) (Chebyshev Inequality) For every \(\epsilon > 0\) and \(f \in L_1(\mu)\), show that
\[
\mu(\{x \in X : |f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X |f| \, d\mu < \infty.
\]

(b) Let \(f \in L_1(\mu)\) and let \(M > 0\) such that
\[
\left| \frac{1}{\mu(E)} \int_E f \, d\mu \right| \leq M, \quad \text{for every } E \in S \text{ with } 0 < \mu(E) < \infty.
\]
Show that \(|f(x)| \leq M\), for a.e. \(x(\mu)\).
Q.3. [2 marks]

Let \((X, \mathcal{S})\) be a measurable space and \(f : X \to \mathbb{R}\) be \(\mathcal{S}\)-measurable. Let \(\mathcal{S}_0 = \{f^{-1}(E) : E \in \mathcal{B}_{\mathbb{R}}\}\). We know that \(\mathcal{S}_0 \subseteq \mathcal{S}\) and \(\mathcal{S}_0\) is a \(\sigma\)-algebra on \(X\). Prove the following.

(a) If \(\phi : \mathbb{R} \to \mathbb{R}\) is Borel measurable, that is, \(\phi^{-1}(E) \in \mathcal{B}_{\mathbb{R}}\), for every \(E \in \mathcal{B}_{\mathbb{R}}\), then \(\phi \circ f : X \to \mathbb{R}\) is \(\mathcal{S}_0\)-measurable.

(b) If \(\psi : X \to \mathbb{R}\) is \(\mathcal{S}_0\)-measurable, then there exists a Borel measurable function \(\phi : \mathbb{R} \to \mathbb{R}\) such that \(\psi = \phi \circ f\).

**Hint.** Use the simple function technique and note that if \(\psi\) is a simple \(\mathcal{S}_0\)-measurable function, then \(\psi = \sum_{i=1}^{n} a_i \chi_{f^{-1}(E_i)}\), for some \(n \in \mathbb{N}\), \(a_i \in \mathbb{R}\) and \(E_i \in \mathcal{R}_{\mathbb{R}}\), for \(i = 1, \ldots, n\). Then show that \(\psi = \left(\sum_{i=1}^{n} a_i \chi_{E_i}\right) \circ f\).