NPTEL Measure Theory, July 2018
Assignment 7

Deadline: Wednesday, September 19, 2018, 11.59 PM

Q.1. **[3 marks]**

Let $(X, \mathcal{S}, \mu)$ be a measure space. Let f, g be nonnegative simple measurable functions.

(a) Show that if $f \geq g$, then $\int_X f \, d\mu \geq \int_X g \, d\mu$.

(b) Let

$$(f \vee g)(x) = \max\{f(x), g(x)\}, \quad (f \wedge g)(x) = \min\{f(x), g(x)\}, \quad \text{for } x \in X.$$

Then show that $f \vee g$ and $f \wedge g$ are nonnegative simple measurable functions. Further show that

$$\int_X (f \wedge g) \, d\mu \leq \int_X f \, d\mu \leq \int_X (f \vee g) \, d\mu,$$

$$\int_X (f \wedge g) \, d\mu \leq \int_X g \, d\mu \leq \int_X (f \vee g) \, d\mu.$$

Q.2. **[4 marks]**

Let $(X, \mathcal{S}, \mu)$ be a measure space. Let (f_n), (g_n) be sequences of nonnegative simple measurable functions such that the sequences $(f_n(x))$ and $(g_n(x))$ are increasing, for all $x \in X$. Suppose

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} g_n(x), \quad \text{for all } x \in X.$$

Then show that

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \lim_{n \to \infty} \int_X g_n \, d\mu.$$

Hint. For all fixed $k \in \mathbb{N}$, consider the sequence $(f_n \wedge g_k)$ to deduce that

$$\int_X g_k \, d\mu \leq \lim_{n\to\infty} \int_X f_n \, d\mu, \quad \text{for all } k \in \mathbb{N}.$$

Q.3. **[3 marks]**

Let $(X, \mathcal{S}, \mu)$ be a measure space. Let (f_n) and (g_n) be sequences of measurable functions such that

$$|f_n| \leq g_n, \quad \text{for all } n \in \mathbb{N}.$$

Let f, g be measurable functions such that

$$\begin{aligned} f_n(x) &\to f(x) \quad \text{for a.e. } x(\mu), \\ g_n(x) &\to g(x) \quad \text{for a.e. } x(\mu). \end{aligned}$$

If

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu < +\infty,$$

then show that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Hint. Apply Fatou's Lemma to the sequences $(g_n - f_n)$ and $(g_n + f_n)$.

Note. Check the Announcement page on the course portal for details on submitting the assignment solutions.