NPTEL Measure Theory, July 2018

Deadline: Wednesday, September 5, 2018, 11.59 PM

Q.1. [2 marks]
Let $\mathcal{I}_o$ be the collection of all open intervals in $\mathbb{R}$. Let λ be the length function for intervals in $\mathbb{R}$. Show that for $E \subseteq \mathbb{R}$,

$$\lambda^*(E) = \inf\left\{ \sum_{i=1}^{\infty} \lambda(I_i) \mid I_i \in \mathcal{I}_o,\ I_i \cap I_j = \emptyset \text{ for } i \neq j \text{ and } E \subseteq \bigcup_{i=1}^{\infty} I_i \right\}.$$

Hint. Let $\tilde{\mathcal{I}}$ be the collection of all left-closed right-open intervals in $\mathbb{R}$. Then we know that for any $E \subseteq \mathbb{R}$,

$$\begin{aligned}
\lambda^*(E) &= \inf\left\{ \sum_{i=1}^{\infty} \lambda(I_i) \mid I_i \in \tilde{\mathcal{I}},\ E \subseteq \bigcup_{i=1}^{\infty} I_i \right\} \\
&= \inf\left\{ \sum_{i=1}^{\infty} \lambda(I_i) \mid I_i \in \tilde{\mathcal{I}},\ I_i \cap I_j = \emptyset, \text{ for } i \neq j \text{ and } E \subseteq \bigcup_{i=1}^{\infty} I_i \right\}.
\end{aligned}$$

Q.2. [6 marks]
We say $E \subseteq \mathbb{R}$ is a (Lebesgue) null set if $\lambda^*(E) = 0$. Prove the following.

(a) Any countably infinite set is a null set.

(b) Every subset of a null set is a null set.

(c) Let A_n, $n \in \mathbb{N}$ be null sets. Then $\bigcup_{n=1}^{\infty} A_n$ is a null set.

(d) Let $E \subseteq [a, b]$ have only finite number of limit points. Can E be uncountable? Is E a null set?

(e) Let E be a null set and $x \in \mathbb{R}$. What can you say about the sets $E + x = \{y + x \mid y \in E\}$ and $xE = \{xy \mid y \in E\}$?

(f) Let I be an interval having atleast two distinct points. Show that I is not a null set.

Q.3. [2 marks]

Let $E \subseteq \mathbb{R}$ be Lebesgue measurable and $x \in \mathbb{R}$. Show that $E + x$ and xE (as defined in Q.2.) are Lebesgue measurable and compute $\lambda(E + x)$ and $\lambda(xE)$ in terms of $\lambda(E)$.

Note. Check the Announcement page on the course portal for details on submitting the assignment solutions.