NPTEL Measure Theory, July 2018
Assignment 4

Deadline: Wednesday, September 5, 2018, 11.59 PM

Q.1. [3 marks]
Let $\mathcal{A} = \{A \subseteq \mathbb{R} \mid \text{either } A \text{ or } \mathbb{R} \setminus A \text{ is countable}\}$. Show that $\mathcal{A}$ is an algebra.

For $A \in \mathcal{A}$, let $\mu(A) = 0$ if A is countable and let $\mu(A) = 1$ if $\mathbb{R} \setminus A$ is countable. Show that μ is a measure on $\mathcal{A}$.

Let μ^* be the outer measure on $\mathcal{P}(\mathbb{R})$ induced by μ. What are all the μ^*-measurable sets?

Q.2. [3 marks]
Let X be a nonempty set and let $\mathcal{A}$ be any algebra of subsets of X. Let $x_0 \in X$ be fixed. For $A \in \mathcal{A}$, define

$$\mu(A) = \begin{cases} 0, & x_0 \notin A \\ 1, & x_0 \in A \end{cases}$$

Show that μ is countably additive. Let μ^* be the outer measure induced by μ. Show that μ^* is either 0 or 1, for every $A \subseteq X$, and $\mu^*(A) = 1$, if $x_0 \in A$.

Q.3. Let $\mathcal{A}$ be a σ-algebra of subsets of a set X (that is an algebra which is also closed under countable unions) and $\mu : \mathcal{A} \to [0, \infty]$ be a measure. For any sequence (E_n) in $\mathcal{A}$, show that

(a) [2 marks]
$\mu(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \mu(E_n)$.

(b) [2 marks]
$\mu(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} \mu(E_n)$.

Hint. Note that

$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k, \quad \limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k,$$

and $\liminf_{n\to\infty} E_n \subseteq \limsup_{n\to\infty} E_n$.

Note. Check the Announcement page on the course portal for details on submitting the assignment solutions.