NPTEL Measure Theory, July 2018
Assignment 2

Deadline: Wednesday, August 15, 2018, 11.59 PM

Assume the following definitions before attempting the problems.

Definitions.

Let $\mathcal{C}$ be a class of subsets of a set X. Consider a set function $\mu : \mathcal{C} \to [0, \infty]$.

(1) We say μ is monotone if

$$\mu(A) \leq \mu(B), \quad \text{for all } A, B \in \mathcal{C},\ A \subseteq B.$$

(2) We say μ is finitely additive if

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i),$$

for all $A_1, \ldots, A_n \in \mathcal{C}$ such that $A_i \cap A_j = \emptyset$, for all $i \neq j$ and $\bigcup_{i=1}^{n} A_i \in \mathcal{C}$.

(3) We say μ is countably additive if

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

for all $A_n \in \mathcal{C}$, $n \in \mathbb{N}$ such that $A_i \cap A_j = \emptyset$, for all $i \neq j$ and $\bigcup_{i=n}^{\infty} A_n \in \mathcal{C}$.

(4) We say μ is countably subadditive if

$$\mu\left(\bigcup_{i=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

for all $A_n \in \mathcal{C}$, $n \in \mathbb{N}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$.

(5) We say μ is a measure on $\mathcal{C}$ if $\emptyset \in \mathcal{C}$ and μ is countably additive.

Problems.

Q.1. Let $\mathcal{C}$ be a collection of subsets of a set X and $\mu : \mathcal{C} \to [0, \infty]$ be measure. State whether the following statements are TRUE or FALSE. If TRUE, prove the statement and if FALSE, give a counterexample.

(a) μ is finitely additive.

(b) μ is monotone.

(c) μ is countably subadditive.

Q.2. Let $\mathcal{C}$ be a semi-algebra of subsets of a set X and $\mu : \mathcal{C} \to [0, \infty]$ be a set function. Then μ is countably subadditive if and only if

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

for all $A \in \mathcal{C}$ and $A_n \in \mathcal{C}$, $n \in \mathbb{N}$ such that $A \subseteq \bigcap_{n=1}^{\infty} A_n$.

Q.3. Let $\mathcal{C}$ be an algebra of subsets of a set X and $\mu : \mathcal{C} \to [0, \infty]$ be finitely additive such that $\mu(X) < \infty$. Show that the following statements are equivalent.

(a) μ is countably additive.

(b)

$$\lim_{n \to \infty} \mu(A_n) = 0,$$

for all $A_n \in \mathcal{C}$, $n \in \mathbb{N}$ such that $A_{n+1} \subseteq A_n$, for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Q.4. Let X be a countably infinite set and let

$$\mathcal{C} = \{\{x\} : x \in X\}.$$

Show that the algebra generated by $\mathcal{C}$ is

$$\mathcal{F}(\mathcal{C}) = \{A \subseteq X : A \text{ or } X \setminus A \text{ is finite}\}.$$

Let $\mu : \mathcal{F}(\mathcal{C}) \to [0, \infty)$ be defined as

$$\mu(A) = \begin{cases} 0, & A \text{ is finite} \\ 1, & X \setminus A \text{ is finite} \end{cases}$$

Show that μ is finitely additive but not countably additive. If X is an uncountable set, show that μ is also countably additive.

Q.5. Let $\mathcal{U}$ be a class of subsets of a nonempty set X such that

(a) $\emptyset \notin \mathcal{U}$.

(b) If $A \in \mathcal{U}$ and $B \supseteq A$, then $B \in \mathcal{U}$.

(c) If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.

(d) For every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Define $\mu : \mathcal{P}(X) \to [0, \infty)$ as

$$\mu(A) = \begin{cases} 1, & A \in \mathcal{U} \\ 0, & A \notin \mathcal{U} \end{cases}$$

Show that μ is finitely additive.

Note. Check the Announcement page on the course portal for details on submitting the assignment solutions.