NPTEL Measure Theory, July 2018
Assignment 1

Deadline: Wedenesday, 15 August, 2018, 11.59 PM

Q.1. [2 marks]
Let $\mathcal{F}$ be any collection of subsets of a set X. Show that $\mathcal{F}$ is an algebra if and only if the following hold.

(a) $\emptyset, X \in \mathcal{F}$.

(b) $X \setminus A \in \mathcal{F}$, for every $A \in \mathcal{F}$.

(c) $A \cup B \in \mathcal{F}$, for all $A, B \in \mathcal{F}$.

Q.2. Let $\mathcal{F}$ be an algebra of subsets of a set X. Show that

(a) [2 marks]
If $A, B \in \mathcal{F}$, then $A \Delta B := (A \setminus B) \cup (B \setminus A) \in \mathcal{F}$.

(b) [3 marks]
If $E_1, \ldots, E_n \in \mathcal{F}$, then there exist $F_1, \ldots, F_n \in \mathcal{F}$ such that

(i) $F_i \subseteq E_i$, for every i.

(ii) $F_i \cap F_j = \emptyset$, for all $i \neq j$.

(iii) $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$.

Q.3. Let $\{\mathcal{F}_\alpha\}_{\alpha \in \Lambda}$ be a family of subsets of a set X and $\mathcal{F} = \bigcap_{\alpha \in \Lambda} \mathcal{F}_\alpha$. State whether each of the following statements is TRUE or FALSE. If TRUE, prove the statement and if FALSE, give a counterexample.

(a) [2 marks]
If $\mathcal{F}_\alpha$ is a semi-algebra, for all $\alpha \in \Lambda$, then $\mathcal{F}$ is a semi-algebra.

(b) [2 marks]
If $\mathcal{F}_\alpha$ is an algebra, for all $\alpha \in \Lambda$, then $\mathcal{F}$ is an algebra.

(c) [2 marks]
If $\mathcal{F}_\alpha$ is a σ-algebra, for all $\alpha \in \Lambda$, then $\mathcal{F}$ is a σ-algebra.

Q.4. Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a family of subsets of a set X and $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. State whether each of the following statements is TRUE or FALSE. If TRUE, prove the statement and if FALSE, give a counterexample.

(a) [1 mark]
If $\mathcal{F}_n$ is a semi-algebra, for all $n \in \mathbb{N}$, then $\mathcal{F}$ is a semi-algebra.

(b) [1 mark]
If $\mathcal{F}_n$ is an algebra, for all $n \in \mathbb{N}$, then $\mathcal{F}$ is an algebra.

(c) [1 mark]
If $\mathcal{F}_n$ is a σ-algebra, for all $n \in \mathbb{N}$, then $\mathcal{F}$ is a σ-algebra.

Q.5. [2 marks]
Let $\mathcal{C}$ be any semi-algebra of subsets of a set X. Show that $\mathcal{F}(\mathcal{C})$, the algebra generated by $\mathcal{C}$, is given by

$$\mathcal{F}(\mathcal{C}) = \left\{ E \subseteq X : E = \bigcup_{i=1}^{n} C_i;\ C_i \in \mathcal{C};\ C_i \cap C_j = \emptyset,\ \text{for } i \neq j;\ n \in \mathbb{N} \right\}.$$

Q.6. [2 marks]
Let X be any nonempty set and

$$\mathcal{C} = \{\{x\} : x \in X\} \cup \{\emptyset, X\}.$$

Is $\mathcal{C}$ a semi-algebra of subsets of X? What is the algebra generated by $\mathcal{C}$? Does the answer depend on whether X is finite or not?