NPTEL Measure Theory, July 2018
Assignment 12

Deadline: Wednesday, October 24, 2018, 23:59 IST

Q.1. **[2 marks]**

Let $f \in L_1(\mathbb{R}^2, \mathcal{L}_{\mathbb{R}^2}, \lambda_{\mathbb{R}^2})$. For $x \in \mathbb{R}^2$, define $\phi_x : \mathbb{R}^2 \to \mathbb{R}$ as

$$\phi_x(y) = \int_{\mathbb{R}^2} f(x+y)\, d\lambda_{\mathbb{R}^2}(x), \quad \text{for all } y \in \mathbb{R}^2.$$

Show that for all $x \in \mathbb{R}^2$, ϕ_x is integrable and further.

$$\int_{\mathbb{R}^2} f(x+y)\, d\lambda_{\mathbb{R}^2}(x) = \int_{\mathbb{R}^2} f(x)\, d\lambda_{\mathbb{R}^2}(x).$$

Q.2. **[6 marks]**

Let $E \in \mathcal{L}_{\mathbb{R}^2}$ and $p = (x, y) \in \mathbb{R}^2$. Let $pE = \{(xs, yt) : (s,t) \in E\}$. Prove the following.

(a) $pE \in \mathcal{L}_{\mathbb{R}^2}$ and $\lambda_{\mathbb{R}^2}(pE) = |xy|\lambda_{\mathbb{R}^2}(E)$, for every $p \in \mathbb{R}^2$ and $E \in \mathcal{L}_{\mathbb{R}^2}$.

(b) For any nonnegative Borel measurable $f : \mathbb{R}^2 \to \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$,

$$\int_{\mathbb{R}^2} f(xs, yt)\, d\lambda_{\mathbb{R}^2}(s,t) = |xy| \int_{\mathbb{R}^2} f(s,t)\, d\lambda_{\mathbb{R}^2}(s,t).$$

(c) Define $\pi := \lambda_{\mathbb{R}^2}\big(\{x \in \mathbb{R}^2 : |x| \le 1\}\big)$. Then

$$\lambda_{\mathbb{R}^2}\big(\{x \in \mathbb{R}^2 : |x| < 1\}\big) = \pi \quad \text{and} \quad \lambda_{\mathbb{R}^2}\big(\{x \in \mathbb{R}^2 : |x| < r\}\big) = \pi r^2, \quad r > 0.$$

(d) Let E be a vector subspace of $\mathbb{R}^2$. Then $\lambda_{\mathbb{R}^2}(E) = 0$, if $\dim E < 2$.

Q.3. **[2 marks]**

Let $E \in \mathcal{L}^2_{\mathbb{R}}$ with $0 < \lambda(E) < \infty$. Show that there exists a unique $(c, d) \in \mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} x\, \chi_E(x + c, y + d)\, d\lambda_{\mathbb{R}^2}(x, y) = 0 = \int_{\mathbb{R}^2} y\, \chi_E(x + c, y + d)\, d\lambda_{\mathbb{R}^2}(x, y).$$

In fact,

$$c = \frac{1}{\lambda(E)} \int_{\mathbb{R}^2} x\, \chi_E(x, y)\, d\lambda_{\mathbb{R}^2}(x, y), \quad d = \int_{\mathbb{R}^2} \frac{1}{\lambda(E)} y\, \chi_E(x, y)\, d\lambda_{\mathbb{R}^2}(x, y).$$

Here, (c, d) is called the centroid of E.

Note. Check the Announcement page on the course portal for details on submitting the assignment solutions.