NPTEL Measure Theory, July 2018
Assignment 11

Deadline: Wednesday, October 17, 2018, 23:59 IST

Q.1. **[2 marks]**

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and let $f : X \times Y \to \mathbb{R}$ be a nonnegative $\mathcal{A} \otimes \mathcal{B}$-measurable function. Let μ be a σ-finite measure on $(Y, \mathcal{B})$. For any $E \in \mathcal{B}$ and $x \in X$, let

$$\eta(x, E) = \int_E f(x, y)\, d\mu(y).$$

For any $E \in \mathcal{B}$, define $\eta^E : X \to \mathbb{R}$ as $\eta^E(x) = \eta(x, E)$, for all $x \in X$.

For any $x \in X$, define $\eta_x : \mathcal{B} \to \mathbb{R}$ as $\eta_x(E) = \eta(x, E)$, for all $E \in \mathcal{B}$.

Show that

(a) η^E is an $\mathcal{A}$-measurable function, for every $E \in \mathcal{B}$.

(b) η_x is a measure on $(Y, \mathcal{B})$, for every $x \in X$.

Q.2. **[2 marks]**

Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be measure spaces and let $f \in L_1(X, \mathcal{A}, \mu)$, $g \in L_1(Y, \mathcal{B}, \nu)$. Define $\phi : X \times Y \to \mathbb{R}$ as

$$\phi(x, y) = f(x)g(y), \quad x \in X,\, y \in Y.$$

Show that $\phi \in L_1(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$. Further show that

$$\int_{X\times Y} \phi\, d(\mu \times \nu) = \left(\int_X f\, d\mu\right)\left(\int_Y g\, d\nu\right).$$

Q.3. **[1 mark]**

Let $f \in L_1(0,a)$ and

$$g(x) = \int_x^a \frac{f(t)}{t}\, d\lambda(t), \quad 0 < x < a.$$

Show that $g \in L_1(0,a)$ and compute $\int_0^a g(x)\, d\lambda(x)$.

Q.4. **[5 marks]**

Let $(X, \mathcal{A})$ be a σ-finite measure space. For any nonnegative function $f : X \to \mathbb{R}$, let

$$\begin{aligned} E^*(f) &= \{(x,t) \in X \times \mathbb{R} : 0 \le t \le f(x)\}, \\ E_*(f) &= \{(x,t) \in X \times \mathbb{R} : 0 \le t < f(x)\}. \end{aligned}$$

Prove the following.

(a) If f_n, $n \in \mathbb{N}$ and f are nonnegative functions on X such that

$$0 \le f_1 \le f_2 \le f_3 \le \cdots \quad \text{and} \quad f_n \to f \text{ pointwise on } X,$$

then

$$E_*(f_1) \subseteq E_*(f_2) \subseteq E_*(f_3) \subseteq \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} E_*(f_n) = E_*(f).$$

(b) If $f : X \to \mathbb{R}$ is a nonnegative measurable function, then $E_*(f), E^*(f) \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$.
Hint. Use (a) and the simple function technique. Further, note that $E^*(f) = \bigcap_{n=1}^{\infty} E_*\big(f + \frac{1}{n}\big)$.

(c) If $f : X \to \mathbb{R}$ is a nonnegative function such that $E^*(f) \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$, then f is measurable.
Hint. For any $c \in \mathbb{R}$, define

$$A_c = \{(x,t) \in X \times \mathbb{R} : f(x) > c,\ t > 0\}.$$

Show that $A_c \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$ by proving the equality

$$A_c = \bigcup_{n=1}^{\infty} \left\{ (x,t) \in X \times \mathbb{R} : \left(x, \frac{t}{n} + c \right) \in E^*(f),\ t > 0 \right\}.$$

(d) Let $f : X \to \mathbb{R}$ be measurable. Then $G(f) \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$, where

$$G(f) = \big\{(x, t) \in X \times \mathbb{R} : f(x) = t\big\}.$$

The set $G(f)$ is called the graph of f.

(e) Let $f \in L_1(X, \mathcal{A}, \mu)$. Then

$$(\mu \times \nu)\big(E^*|f|\big) = \int_X |f| \, d\mu = (\mu \times \nu)\big(E_*|f|\big).$$

Hint. First prove for the case when f is bounded and $\mu(X) < \infty$. Then prove for general f with $\mu(X) < \infty$, by considering $f_n = \min\{|f|, n\}$, $n \in \mathbb{N}$ and using (a). Then finally extend the result for the case when μ is σ-finite.