NPTEL Measure Theory, July 2018
Assignment 10

Deadline: Wednesday, October 10, 2018, 23:59 IST

Q.1. **[2 marks]**

Let $(X, \mathcal{A})$ be a measurable space. Let $\alpha, \beta \in \mathbb{R}$ and $E \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$. Let

$$E_{\alpha,\beta}\{(x,t) \in X \times \mathbb{R} : (x, \alpha t + \beta) \in E\}.$$

Show that $E_{\alpha,\beta} \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$.

Hint. Use the σ-algebra technique.

Q.2. **[1 mark]**

Let $E \in \mathcal{B}_{\mathbb{R}}$. Let

$$E^+ = \{(x,y) \in \mathbb{R}^2 : x + y \in E\} \quad \text{and} \quad E^- = \{(x,y) \in \mathbb{R}^2 : x - y \in E\}.$$

Show that E^+, $E^- \in \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

Q.3. **[2 marks]**

Let $\mathcal{B}_{\mathbb{R}^2}$ denote the σ-algebra of Borel subsets of $\mathbb{R}^2$, that is, the σ-algebra generated by open subsets of $\mathbb{R}^2$. Show that $\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

Q.4. **[4 marks]**

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. For $E, F \in \mathcal{A} \otimes \mathcal{B}$ and $E_i \in \mathcal{A} \otimes \mathcal{B}$, $i \in I$, where I is any indexing set, prove the following.

(a) $\left(\bigcup_{i \in I} E_i\right)_x = \bigcup_{i \in I} (E_i)_x$ and $\left(\bigcup_{i \in I} E_i\right)^y = \bigcup_{i \in I} (E_i)^y$.

(b) $\left(\bigcap_{i \in I} E_i\right)_x = \bigcap_{i \in I} (E_i)_x$ and $\left(\bigcap_{i \in I} E_i\right)^y = \bigcap_{i \in I} (E_i)^y$.

(c) $(E \setminus F)_x = E_x \setminus F_x$ and $(E \setminus F)^y = E^y \setminus F^y$.

(d) If $E \subseteq F$, then $E_x \subseteq F_x$ and $E^y \subseteq F^y$.

(e) If $\mu(E^y) = 0$, for a.e. $y \in Y$ (ν), then show that $\nu(E_x) = 0$, for a.e. $x \in X$ (μ). What can you say about $(\mu \times \nu)(E)$?