NPTEL Measure Theory, July 2018
Assignment 0

Note. This is just an introductory assignment. Your submission will be graded, but it will not count towards your final score in the course. You are encouraged to try all the problems to the best of your knowledge. It is completely fine if you are unable to solve some problem. You will also find some light discussion going on along with the problems.

1. Let X be a nonempty set. A nonempty family $\mathcal{C}$ of subsets of X is called a partition of X if

(a) $\bigcup_{A\in\mathcal{C}} A = X$.

(b) $A \cap B = \emptyset$, for all $A, B \in \mathcal{C}$, $A \neq B$.

A few examples and nonexamples are as follows.

(a) $\{\{2,4,6,\ldots\},\{1,3,5,\ldots\}\}$ is a partition of the set of all natural numbers $\{1,2,3,\ldots\}$.

(b) For a nonempty set X, $\{\emptyset, X\}$ is a partition of X.

(c) For a nonempty set X, $\{\{x\} : x \in X\}$ is a partition of X.

(d) $\{\{1,2,3,4\}\}$ is a partition of $\{1,2,3,4\}$.

(e) $\{\{1\},\{1,2\},\{1,2,3\}\}$ is NOT a partition of $\{1,2,3\}$.

(f) $\{\emptyset\} \cup \{\{x\} : x \in \mathbb{R}, x > 0\}$ is NOT a partition of the set $\mathbb{R}$ of real numbers.

Question. Consider the set of natural numbers $\mathbb{N} = \{1,2,3,\ldots\}$. For any $n \in \mathbb{N}$, define

$$\langle n\rangle = \{m \in \mathbb{N} : |n-m| \text{ is a multiple of } 3\}.$$

Then show that $\mathcal{C} = \{\langle n\rangle : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ and $|\mathcal{C}| = 3$.

2. Let X be a nonempty set. A subset $E \subseteq X \times X$ is called an equivalence relation on X if

(a) $(x, x) \in E$, for all $x \in X$.

(b) For any $x, y \in X$, if $(x, y) \in E$, then $(y, x) \in E$.

(c) For any $x, y, z \in X$, if $(x, y), (y, z) \in E$, then $(x, z) \in E$.

For any $x, y \in X$, if $(x, y) \in E$, then we denote it as $x \sim_E y$ or $x \sim y$. Thus we say $\sim$ is an equivalence relation on X and we have

(a) $x \sim x$, for all $x \in X$.

(b) For any $x, y \in X$, if $x \sim y$, then $y \sim x$.

(c) For any $x, y, z \in X$, if $x \sim y, y \sim z$, then $x \sim z$.

For any $x \in X$, the set

$$\langle x \rangle_\sim = \{y \in X : x \sim y\}$$

is called the equivalence class of x under $\sim$.

In **Question 1.**, for any $n \in \mathbb{N}$, the set $\langle n \rangle$ is the equivalence class of n under the relation $\sim$ on $\mathbb{N}$, which is defined by $n \sim m$ if $|n - m|$ is a multiple of 3.

Question (a). Let X be a nonempty set and let $\sim$ be an equivalence relation on X. Then show that the family

$$\mathcal{C}_\sim = \{\langle x \rangle_\sim : x \in X\}$$

is a partition of X.

This explains why $\mathcal{C}$ was a partition of $\mathbb{N}$ in **Question 1.**

Question (b). Let X be a nonempty set and let $\mathcal{C}$ be a partition of X consisting of nonempry subsets of X. Then show that there exists an equivalence relation $\sim$ on X such that $\mathcal{C} = \mathcal{C}_\sim$.

Hint. Define $\sim$ on X by $x \sim y$ if there exists $A \in \mathcal{C}$ such that $x, y \in A$.

3. The Archimedian Property states that for any $x \in \mathbb{R}, x > 0$, there exists $n \in \mathbb{N}$ such that $x < n$.

Question. Show that given any $a, b \in \mathbb{Q}$, $a < b$, there exist $r \in \mathbb{Q}$, $s \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$ and $a < s < b$.

4. Let I be an open interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be continuous at $p \in I$ if for every sequence (p_n) of points in I such that $p_n \to p$, we have $f(p_n) \to f(p)$. Further, we say f is continuous on I if f is continuous at p, for all $p \in I$. If f is not continuous at $p \in I$, then we say f is discontinuous at p.

Question. Let $f : (0, 1) \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that f is discontinuous at every $x \in (0, 1)$.

5. Question. For integers $a, b \in \mathbb{Z}$, $a \leq b$, define

$$\langle a, b \rangle = \{k \in \mathbb{Z} : a \leq k \leq b\}.$$

Note that $|\langle a, b \rangle| = b - a + 1$, for all $a, b \in \mathbb{Z}$, $a \leq b$. Further, for any $a_i \leq b_i$ in $\mathbb{Z}$, $i = 1, \ldots, n$, we have

$$|\langle a_1, b_1 \rangle \times \cdots \times \langle a_n, b_n \rangle| = |\langle a_1, b_1 \rangle| \times \cdots \times |\langle a_n, b_n \rangle|.$$

Find $\big|\big(\langle 1, 10 \rangle \times \langle 2, 8 \rangle\big) \setminus \big(\langle 3, 7 \rangle \times \langle 4, 5 \rangle\big)\big|$.