1)a)

First, we construct the two-period binomial tree for the stock price.

```
Year 0          Year 1          Year 2
          25.680
     20          22.1028
          17.214
```

The calculations for the stock prices at various nodes are as follows:

\[
S_u = 20 \times 1.2840 = 25.680 \\
S_d = 20 \times 0.8607 = 17.214 \\
S_{uw} = 25.68 \times 1.2840 = 32.9731 \\
S_{ud} = S_u = 17.214 \times 1.2840 = 22.1028 \\
S_{dd} = 17.214 \times 0.8607 = 14.8161
\]

The risk-neutral probability for the stock price to go up is

\[
P^* = \frac{e^{r_d} - d}{u - d} = \frac{e^{0.05} - 0.8607}{1.2840 - 0.8607} = 0.4502.
\]

Thus, the risk-neutral probability for the stock price to go down is 0.5498.

If the option is exercised at time 2, the value of the call would be

\[
C_{uw} = (32.9731 - 22)_\text{u} = 10.9731 \\
C_{ud} = (22.1028 - 22)_\text{d} = 0.1028 \\
C_{dd} = (-14.8161 - 22)_\text{d} = 0
\]

If the option is European, then \( C_u = e^{-0.05} [0.4502 C_{uw} + 0.5498 C_{ud}] = 4.7530 \) and \( C_d = e^{-0.05} [0.4502 C_{ud} + 0.5498 C_{dd}] = 0.0440 \).

But since the option is American, we should compare \( C_u \) and \( C_d \) with the value of the option if it is exercised at time 1, which is 3.68 and 0, respectively. Since 3.68 < 4.7530 and 0 < 0.0440, it is not optimal to exercise the option at time 1 whether the stock is in the up or down state. Thus the value of the option at time 1 is either 4.7530 or 0.0440.

Finally, the value of the call is

\[
C = e^{-0.05} [0.4502(4.7530) + 0.5498(0.0440)] = 2.0583.
\]
1)b)

(a) The probability distributions are:
\[ \phi(2.0 + 0.1, 0.16) = \phi(2.1, 0.16) \]
\[ \phi(2.0 + 0.6, 0.16 \times 6) = \phi(2.6, 0.96) \]
\[ \phi(2.0 + 1.2, 0.16 \times 12) = \phi(3.2, 1.96) \]

(b) The chance of a random sample from \( \phi(2.6, 0.96) \) being negative is
\[ N \left(-\frac{2.6}{\sqrt{0.96}}\right) = N(-2.65) \]

where \( N(x) \) is the cumulative probability that a standardized normal variable [i.e., a variable with probability distribution \( \phi(0, 1) \)] is less than \( x \). From normal distribution tables \( N(-2.65) = 0.0040 \). Hence the probability of a negative cash position at the end of six months is 0.40%.

Similarly the probability of a negative cash position at the end of one year is
\[ N \left(-\frac{3.2}{\sqrt{1.96}}\right) = N(-2.30) = 0.0107 \]
or 1.07%.

(c) In general the probability distribution of the cash position at the end of \( x \) months is
\[ \phi(2.0 + 0.1x, 0.16x) \]

The probability of the cash position being negative is maximized when:
\[ \frac{2.0 + 0.1x}{\sqrt{0.16x}} \]
is minimized. Define
\[ y = \frac{2.0 + 0.1x}{0.4\sqrt{x}} = 5x^{-\frac{1}{2}} + 0.25x^{\frac{1}{2}} \]

\[ \frac{dy}{dx} = -2.5x^{-\frac{3}{2}} + 0.125x^{-\frac{1}{2}} \]
\[ = x^{-\frac{1}{2}}(-2.5 + 0.125x) \]
2)a)  

The process followed by $B$, the bond price, is from Itô's lemma:

$$
 dB = \left[ \frac{\partial B}{\partial x} a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2} \frac{\partial^2 B}{\partial x^2} s^2 x^2 \right] dt + \frac{\partial B}{\partial x} s dx
$$

In this case

$$
 B = \frac{1}{x}
$$

so that:

$$
 \frac{\partial B}{\partial t} = 0, \quad \frac{\partial B}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial^2 B}{\partial x^2} = \frac{2}{x^3}
$$

Hence

$$
 dB = \left[ -a(x_0 - x) \frac{1}{x^2} + \frac{1}{2} \frac{s^2 x^2}{x^2} \right] dt - \frac{1}{x^2} s dx
$$

$$
 = -a(x_0 - x) \frac{1}{x^2} dt - \frac{s}{x} dx
$$

The expected instantaneous rate at which capital gains are earned from the bond is therefore:

$$
 -a(x_0 - x) \frac{1}{x^2} + \frac{s^2}{x}
$$

The expected interest per unit time is $1$. The total expected instantaneous return is therefore:

$$
 1 - a(x_0 - x) \frac{1}{x^2} + \frac{s^2}{x}
$$

When expressed as a proportion of the bond price this is:

$$
 \left( 1 - a(x_0 - x) \frac{1}{x^2} + \frac{s^2}{x} \right) / \left( \frac{1}{x} \right)
$$

$$
 = x - \frac{a}{x} (x_0 - x) + \frac{s^2}{x}
$$

2)b)  

A stock price is currently 50. Its expected return and volatility are 12% and 30%, respectively. What is the probability that the stock price will be greater than 80 in two years? (Hint $S_T > 80$ when in $\ln S_T > \ln 80$.)

The variable $\ln S_T$ is normally distributed with mean $\ln S_0 + (\mu - \sigma^2/2)T$ and standard deviation $\sigma \sqrt{T}$. In this case $S_0 = 50$, $\mu = 0.12$, $T = 2$, and $\sigma = 0.30$ so that the mean and standard deviation of $\ln S_T$ are $\ln 50 + (0.12 - 0.30^2/2)2 = 4.062$ and $0.3 \sqrt{2} = 0.424$, respectively. Also, $\ln 80 = 4.382$. The probability that $S_T > 80$ is the same as the probability that $\ln S_T > 4.382$. This is

$$
 1 - N \left( \frac{4.382 - 4.062}{0.424} \right) = 1 - N(0.754)
$$

where $N(x)$ is the probability that a normally distributed variable with mean zero and standard deviation 1 is less than $x$. From the tables at the back of the book $N(0.754) = 0.775$ so that the required probability is 0.225.
3) a)

\[ d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2 \sqrt{0.5}} = 0.7693 \]
\[ d_2 = \frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2 \sqrt{0.5}} = 0.6278 \]

and

\[ Ke^{-rT} = 40e^{-0.05} = 38.049 \]

Hence, if the option is a European call, its value \( c \) is given by

\[ c = 42N(0.7693) - 38.049N(0.6278) \]

If the option is a European put, its value \( p \) is given by

\[ p = 38.049N(-0.6278) - 42N(-0.7693) \]

Using the polynomial approximation just given or the NORMSDIST function in Excel,

\[ N(0.7693) = 0.7791, \quad N(-0.7693) = 0.2209 \]
\[ N(0.6278) = 0.7249, \quad N(-0.6278) = 0.2651 \]

so that

\[ c = 4.76, \quad p = 0.81 \]

Ignoring the time value of money, the stock price has to rise by $2.76 for the purchaser of the call to break even. Similarly, the stock price has to fall by $2.81 for the purchaser of the put to break even.

3) b)

In the case \( c = 2.5, S_0 = 15, K = 13, T = 0.25, r = 0.05 \). The implied volatility must be calculated using an iterative procedure.

A volatility of 0.2 (or 20% per annum) gives \( c = 2.20 \). A volatility of 0.3 gives \( c = 2.32 \). A volatility of 0.4 gives \( c = 2.50 \). A volatility of 0.5 gives \( c = 2.487 \). By interpolation the implied volatility is about 0.397 or 39.7% per annum.
4)a)

\[ D_1 = D_2 = 1.50, \quad \alpha_1 = 0.3333, \quad \alpha_2 = 0.8533, \quad T = 1.25, \quad r = 0.08 \text{ and } K = 50 \]

\[ K \left[ 1 - e^{-r(T-t_0)} \right] = 50 \left( 1 - e^{-0.08 \times 0.1107} \right) = 1.80 \]

Hence:

\[ D_2 < K \left[ 1 - e^{-r(T-t_0)} \right] \]

Also:

\[ K \left[ 1 - e^{-r(\alpha_1-t_0)} \right] = 50 \left( 1 - e^{-0.08 \times 0.5} \right) = 2.16 \]

Hence:

\[ D_1 < K \left[ 1 - e^{-r(\alpha_1-t_0)} \right] \]

It follows from the conditions established in Section 13.12 that the option should never be exercised early.

The present value of the dividends is

\[ 1.5e^{-0.3333 \times 0.08} + 1.5e^{-0.8533 \times 0.08} = 2.864 \]

The option can be valued using the European pricing formula with:

\[ S_0 = 50 - 2.864 = 47.136, \quad K = 50, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25 \]

\[ d_1 = \frac{\ln(47.136/50) + (0.08 + 0.25^2/2) \times 1.25}{0.25 \sqrt{1.25}} = -0.0545 \]

\[ d_2 = d_1 - 0.25 \sqrt{1.25} = -0.3340 \]

\[ N(d_1) = 0.4783, \quad N(d_2) = 0.3092 \]

and the call price is

\[ 47.136 \times 0.4783 - 50 e^{-0.08 \times 1.25} \times 0.3092 = 4.17 \]

or $4.17.

4)b)

In this case \( S_0 = 50, \mu = 0.18 \text{ and } \sigma = 0.30 \). The probability distribution of the stock price in two years, \( S_T \), is lognormal and is, from equation (13.3), given by:

\[ \ln S_T \sim \phi[\ln 50 + (0.18 - \frac{0.09}{2}) \times 0.30^2 \times 2] \]

i.e.,

\[ \ln S_T \sim \phi(4.18, 0.18) \]

The mean stock price is from equation (12.4)

\[ 50 e^{x_2 \times 0.30} = 50 e^{0.09} \times 71.67 \]

and the standard deviation is, from equation (13.5),

\[ 50 e^{x_2 \times 0.30} \sqrt{0.30^2 \times 2} = 31.83 \]

95\% confidence intervals for \( \ln S_T \) are

\[ 4.18 - 1.96 \times 0.42 \text{ and } 4.18 + 1.96 \times 0.42 \]

i.e.,

\[ 3.25 \text{ and } 5.01 \]

These correspond to 95\% confidence limits for \( S_T \) of

\[ 28.52 \text{ and } 150.44 \]

5) The bias is estimated by \( av - 2 \); the square root of \( var \) estimates the standard error; a t statistic for testing the null is the estimated bias divided by the square root of its estimated variance, \( var/4000 \);
and the confidence interval is given by the interval between the 200th and the 3800th r values, adjusted for any bias