Assignment 12

The due date for submitting this assignment has passed.  
Due on 2021-04-14, 23:59 IST.

As per our records you have not submitted this assignment.

Singular values and vectors of a linear map, SVD, 
Polar decomposition and some applications

1) Let $T : V \rightarrow W$ be a linear map. Which of the following statements are necessarily true?  

☐ range($T^*T$) = range($T$)  
☐ dim(range($T^*T$)) = dim(range($T$))  
☐ null($\sqrt{T^*T}$) = null($T$)  
☐ dim(null($T^*$)) = dim(null($T$))

No, the answer is incorrect.  
Score: 0

Accepted Answers:  
dim(range($T^*T$)) = dim(range($T$))  
null($\sqrt{T^*T}$) = null($T$)

2) Let $A = UDV^*$ be the singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \end{bmatrix}$.  

Find $U$, $D$, and $V$.  

Note: $V^*$ is the conjugate transpose of $V$.

1 point
Singular Values and Vectors of a Linear Map

\[ U = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \quad V = \begin{bmatrix} -0.2673 & 0.8944 & -0.3586 \\ -0.5345 & -0.4472 & -0.7171 \\ -0.8018 & 0.0000 & 0.5976 \end{bmatrix}; \quad D = \begin{bmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix} \]

\[ V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; U = \begin{bmatrix} -0.2673 & 0.8944 & -0.3586 \\ -0.5345 & -0.4472 & -0.7171 \\ -0.8018 & 0.0000 & 0.5976 \end{bmatrix}; \quad D = \begin{bmatrix} 0 & \sqrt{14} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \]

\[ U = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}; V = \begin{bmatrix} -0.2673 & 0.8944 & -0.3586 \\ -0.5345 & -0.4472 & -0.7171 \\ -0.8018 & 0.0000 & 0.5976 \end{bmatrix}; \quad D = \begin{bmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix} \]

\[ U = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}; V = \begin{bmatrix} -0.2673 & 0.8944 & -0.3586 \\ -0.5345 & -0.4472 & -0.7171 \\ -0.8018 & 0.0000 & 0.5976 \end{bmatrix}; \quad D = \begin{bmatrix} 0 & \sqrt{14} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \]

No, the answer is incorrect.
Score: 0
Accepted Answers:

\[ U = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; V = \begin{bmatrix} -0.2673 & 0.8944 & -0.3586 \\ -0.5345 & -0.4472 & -0.7171 \\ -0.8018 & 0.0000 & 0.5976 \end{bmatrix}; \quad D = \begin{bmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix} \]

Use below information for the following 2 questions:
Consider a MIMO system with two transmitter and three receiver antennas and following the gain matrix

\[ H = \begin{bmatrix} 1 & 1 \\ i & 1 \\ 1 & i-1 \end{bmatrix}. \]

Let \( z \in \mathbb{C}^3 \) be the noise that gets added in the process i.e. if \( x \in \mathbb{C}^2 \) is the transmitted vector, then the received vector \( y \) is \( y = Hx + z \). Assume that both the transmitter and receiver know \( H \) and can compute SVD, and use the transmission scheme discussed in the lecture. Let \( t = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \) be the message to be sent.

3) Find the vector to be transmitted. (Choose the most appropriate option) 1 point

- \[ \begin{bmatrix} 3i+1 \\ -i-1 \end{bmatrix} \]
- \[ \begin{bmatrix} i-1 \\ -i-1 \end{bmatrix} \]
- \[ \begin{bmatrix} i-2 \\ -i-1 \end{bmatrix} \]
4) Find \( z \) for which \( i \) is retrieved without any error at the receiver. (Note: \( \hat{0} \) denotes the all zero vector.)

- any \( z \) with \( \| z \| \leq |\det(V)| \)
- \( \hat{0} \)
- \( \hat{0} + \null((-1, -1, 1 + i), (1, -i, 1)) \)
- \( \hat{0} + \null(U^*) \)

No, the answer is incorrect.
Score: 0
Accepted Answers:
- \( \hat{0} \)
- \( \hat{0} + \null((-1, -1, 1 + i), (1, -i, 1)) \)
- \( \hat{0} + \null(U^*) \)

No, the answer is incorrect.
Score: 0
Accepted Answers:

5) Let \( T \) be a linear operator on \( \mathbb{R}^4 \) such that \( T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4) \). If \( a \) is the sum of the singular values of \( T \) and \( b \) is the product of the singular values of \( T \). Then, find \( b - a \).

No, the answer is incorrect.
Score: 0
Accepted Answers:
- (Type: Range) -8.01, -7.99

No, the answer is incorrect.
Score: 0
Accepted Answers:
- (Type: Range) 5.58, 5.62

6) Find the sum of the singular values of the differentiation operator \( D \) on \( P_2(\mathbb{R}) \) (the set of all polynomials of degree \( \leq 2 \) with real coefficients) under the inner product \( \langle p(x), q(x) \rangle = \int_{-1}^{1} p(x) q(x) dx \). (Note: \( Dp(x) = p'(x), \forall p(x) \in P_2(\mathbb{R}) \))

No, the answer is incorrect.
Score: 0
Accepted Answers:
- (Type: Range) 5.58, 5.62

7)
Consider the matrix \( A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix} \) with singular values \( \sigma_1 = 3, \sigma_2 = 2 \) and corresponding right singular vectors \( v_1 = (0, 1), v_2 = (-1, 0) \). Which of the following could be left singular vectors of \( A \)?

- (1, 0, 0)
- (0, 1, 0)
- (0, 0, 1)
- (−1, 0, 0)

No, the answer is incorrect.

Score: 0

Accepted Answers:

- (1, 0, 0)
- (0, 1, 0)
- (0, 0, 1)
- (−1, 0, 0)

8) Select the correct statement(s):

- A matrix must be of full rank to have a singular value decomposition.
- All matrices have a singular value decomposition.
- 0 can be a singular value of a matrix.
- Non-zero singular values of an operator \( T \) and its adjoint \( T^* \) are the same.

No, the answer is incorrect.

Score: 0

Accepted Answers:

- All matrices have a singular value decomposition.
- 0 can be a singular value of a matrix.
- Non-zero singular values of an operator \( T \) and its adjoint \( T^* \) are the same.

9) Consider the matrix \( A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \) having the singular value decomposition:

\[
A = UDV = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{18}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & 4 \sqrt{18} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}.
\]

Let \( \{f_1, f_2\} \) be the columns of \( U \), and let \( \{e_1, e_2, e_3\} \) be the rows of \( V \) in the above SVD of \( A \). Select the correct statement(s).

- \( \{f_1, f_2\} \) is a basis for range(\( A \))
\( \{f_1, f_2\} \) is a basis for \( \text{null}(A) \)

\( \{e_1, e_2, e_3\} \) is a basis for \( \text{range}(A) \)

\( \{e_3\} \) is a basis for \( \text{null}(A) \)

No, the answer is incorrect.
Score: 0
Accepted Answers:
\( \{f_1, f_2\} \) is a basis for \( \text{range}(A) \)
\( \{e_3\} \) is a basis for \( \text{null}(A) \)

10) Let \( A \) be a \( 2 \times 2 \) symmetric matrix with eigenvalues \( 3, -2 \) and corresponding eigenvectors \( u_1 \) and \( u_2 \) of unit norm.
\((u_1 \text{ and } u_2 \text{ are column vectors})\)
Select the correct option(s):

- SVD of \( A \) is \[
\begin{bmatrix}
  u_1 & -u_2 \\
\end{bmatrix}
\begin{bmatrix}
  3 & 0 \\
  0 & 2 \\
\end{bmatrix}
\begin{bmatrix}
  u_1^T \\
  u_2^T \\
\end{bmatrix}
\]

- SVD of \( A \) is \[
\begin{bmatrix}
  u_1 & u_2 \\
\end{bmatrix}
\begin{bmatrix}
  3 & 0 \\
  0 & -2 \\
\end{bmatrix}
\begin{bmatrix}
  u_1^T \\
  u_2^T \\
\end{bmatrix}
\]

Polar decomposition of \( A \) is \((u_1 u_1^T + u_2 u_2^T) \cdot (3u_1 u_1^T - 2u_2 u_2^T)\)

Polar decomposition of \( A \) is \((u_1 u_1^T - u_2 u_2^T) \cdot (3u_1 u_1^T + 2u_2 u_2^T)\)

No, the answer is incorrect.
Score: 0
Accepted Answers:

- SVD of \( A \) is \[
\begin{bmatrix}
  u_1 & -u_2 \\
\end{bmatrix}
\begin{bmatrix}
  3 & 0 \\
  0 & 2 \\
\end{bmatrix}
\begin{bmatrix}
  u_1^T \\
  u_2^T \\
\end{bmatrix}
\]

- Polar decomposition of \( A \) is \((u_1 u_1^T + u_2 u_2^T) \cdot (3u_1 u_1^T - 2u_2 u_2^T)\)

11) Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be such that \( T(x, y) = (x + y, y, -x + y) \).
Which of the following could be a Singular Value Decomposition (SVD) of \( T \)? Select the correct option(s).
No, the answer is incorrect.
Score: 0
Accepted Answers:
12) Let \( T : \mathbb{C}^2 \to \mathbb{C}^2 \) be such that \( T(x, y) = (x - iy, (1 - i)x + (1 + i)y) \).

Which of the following could be a polar decomposition for \( T \)?

Select the correct option.

No, the answer is incorrect.

Score: 0

Accepted Answers:

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} i \\
0.5 - 0.5i & 0.5 + 0.5i
\end{pmatrix}
\begin{pmatrix}
1 + \frac{1}{\sqrt{2}} & (1 + \frac{1}{\sqrt{2}})i \\
(1 - \frac{1}{\sqrt{2}})i & 1 + \frac{1}{\sqrt{2}}
\end{pmatrix}
\]