Assignment 10

The due date for submitting this assignment has passed. Due on 2021-03-31, 23:59 IST.
As per our records you have not submitted this assignment.

Self-Adjoint operators, Normal Operators, Spectral theorems

1) Which of the following are normal operators? (operators are represented in standard bases)
   - $T$ over $\mathbb{R}^2$, defined by $T(a, b) = (2a - 2b, 2a + 5b)$
   - $T$ over $\mathbb{C}^2$, defined by $T(a, b) = (2a + ib, a + 2b)$
   - $T$ over $\mathbb{C}^3$, defined by $T(x, y, z) = (2x + iy, -ix + 3y + (1 + i)z, y(1 - i) + 5z)$
   - None of these

No, the answer is incorrect.
Score: 0
Accepted Answers:
   - $T$ over $\mathbb{C}^2$, defined by $T(a, b) = (2a + ib, a + 2b)$
   - $T$ over $\mathbb{C}^3$, defined by $T(x, y, z) = (2x + iy, -ix + 3y + (1 + i)z, y(1 - i) + 5z)$

2) Let $T$ be a normal operator over the vector space $\mathbb{F}^n$. Which of the following are true?
   - If $\mathbb{F}^n$ is $\mathbb{C}^n$, and $T$ is self-adjoint, then all the eigenvalues of $T$ must be real.
   - If $\mathbb{F}^n$ is $\mathbb{C}^n$, then all the eigenvalues of $T$ must be real.
The total sum of the eigenvalues of $T$ and $T^*$ together must be real.

If $v$ is an eigenvector of $T$, then it must also be an eigenvector of $TT^*$

No, the answer is incorrect.
Score: 0
Accepted Answers:
If $\mathbb{F}^n$ is $\mathbb{C}^n$, and $T$ is self-adjoint, then all the eigenvalues of $T$ must be real.
The total sum of the eigenvalues of $T$ and $T^*$ together must be real.
If $v$ is an eigenvector of $T$, then it must also be an eigenvector of $TT^*$

3) Consider a normal operator $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ with eigenvalues $3, 4$ and the vector $v = (3, 2)$ as an eigenvector with the eigenvalue $3$. Which of the following is an eigenvector of $T$ with eigenvalue $4$?

- $(3, 2)$
- $(-2, 3)$
- $(1, 1)$
- $(4, 2)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$(-2, 3)$

4) Let $T$ be a normal operator on an inner product space $V$. If $v, w \in V$ satisfy the equations $Tv = 3v, Tw = 4w$ and $\|v\| = \|w\| = 1, \|T(v + w)\| = $

No, the answer is incorrect.
Score: 0
Accepted Answers:
(Type: Numeric) 5

5) Let $T$ be a normal operator over $\mathbb{C}^3$ with eigenvalues $2, 3$ and $3$. Which of the following statements are true?

- $T$ is diagonal under any orthonormal basis.
- $T$ is diagonal under a non-orthogonal basis of $\mathbb{C}^3$.
- There exist multiple bases which are not simply rearrangements or scaling of vectors from another basis, under which $T$ has a diagonal representation.
- The geometric multiplicity of the eigenvalue $3$ must be equal to $2$.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$T$ is diagonal under any orthonormal basis of $\mathbb{C}^3$.
There exist multiple bases which are not simply rearrangements or scaling of vectors from another basis, under which \(T\) has a diagonal representation.

The geometric multiplicity of the eigenvalue 3 must be equal to 2.

6) Let an operator \(T\) over \(\mathbb{R}^n\) be represented by a matrix \(A\) (under the standard basis) as:

\[
A = uu^T + vv^T
\]

where \(u\) and \(v\) are column vectors that are linearly independent. Which of the following statements are true?

- [ ] If \(\langle u, v \rangle = 0\), then \(u\) and \(v\) are eigenvectors of \(T\)
- [ ] \(A\) is diagonalizable under an orthonormal basis.
- [ ] \(T\) has rank 2.
- [ ] \(u\) is necessarily an eigenvector of \(T\)

No, the answer is incorrect.

Score: 0

Accepted Answers:

- If \(\langle u, v \rangle = 0\), then \(u\) and \(v\) are eigenvectors of \(T\)
- \(A\) is diagonalizable under an orthonormal basis.
- \(T\) has rank 2.

7) Consider the following matrix:

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0.2 - 0.2i \\
0 & 0.2 + 0.2i & 0
\end{bmatrix}
\]

Let the matrix \(B = \lim_{k \to \infty} A^k\).

That is, \(B\) is the matrix we obtain when \(A\) is multiplied to itself repeatedly infinite times. Find the sum of the elements of \(B\).

(For example the sum of the elements of \(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) is \(1 + 2 + 3 + 4 = 10\))

No, the answer is incorrect.

Score: 0

Accepted Answers:

- (Type: Range) 0.99,1.01

8) Let \(V\) be an inner-product space over the field \(\mathbb{C}\).

Consider an operator \(T : V \to V\) represented in the standard basis as:

\[
\begin{bmatrix}
0.5 & 13 + 2i & -9 & 5 & 5 + 3i \\
13 - 2i & 7 & 7 + 9i & 6 & 4 - i \\
-9 & 7 - 9i & 3 & 13 & 6 \\
5 & 6 & 13 & 8 & 1 \\
5 - 3i & 4 + i & 6 & 1 & 13
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.5 & 13 + 2i & -9 & 5 & 5 + 3i \\
13 - 2i & 7 & 7 + 9i & 6 & 4 - i \\
-9 & 7 - 9i & 3 & 13 & 6 \\
5 & 6 & 13 & 8 & 1 \\
5 - 3i & 4 + i & 6 & 1 & 13
\end{bmatrix}
\]

https://onlinecourses.nptel.ac.in/noc21_ee38/unit?unit=42&assessment=115
Let \( v_1 = \begin{bmatrix} 1 \\ 1 + 3i \\ -2 \\ -5 \\ 3 + i \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 5 + 6i \\ 7 \\ 9 + 13i \\ 13 \\ 2 \end{bmatrix} \). (also represented in the standard basis)

Find the imaginary part of \( \langle Tv_1, v_1 \rangle + \langle Tv_2, v_2 \rangle \).

(For example the imaginary part of \( 2 + 3i \) is 3)

No, the answer is incorrect.
Score: 0
Accepted Answers:
(Type: Numeric) 0

9) Let \( V \) be a finite-dimensional vector space over a field \( \mathbb{F} \) and \( T : V \to V \) be an operator on \( V \).

Consider the statement:

\[ \langle Tv, v \rangle = 0 \text{ for all } v \in V \Rightarrow T = 0 \]

For which of the following cases is this statement true? Select the correct option(s).

- \( \mathbb{F} = \mathbb{R} \) and \( T \) is normal
- \( \mathbb{F} = \mathbb{C} \) and \( T \) is normal
- \( \mathbb{F} = \mathbb{R} \) and \( T \) is self-adjoint
- \( \mathbb{F} = \mathbb{C} \) and \( T \) is self-adjoint

No, the answer is incorrect.
Score: 0
Accepted Answers:
\( \mathbb{F} = \mathbb{C} \) and \( T \) is normal
\( \mathbb{F} = \mathbb{R} \) and \( T \) is self-adjoint
\( \mathbb{F} = \mathbb{C} \) and \( T \) is self-adjoint

10) Let \( V \) be a finite-dimensional vector space over \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \) and \( T : V \to V \) be an operator on \( V \).

Select the correct statement(s):

- If \( T \) is self-adjoint, then \( \text{null}(T) = \text{range}(T)^\perp \)
- If \( T \) is normal, then \( \text{null}(T) = \text{range}(T)^\perp \)
- If \( T \) is self-adjoint, then \( \text{range}(T) = \text{range}(T^*) \)
- If \( T \) is normal, then \( \text{null}(T) = \text{null}(T^*) \)

1 point
11) Find the rank-2 approximation of the matrix $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 - i \\ 0 & 1 + i & 0 \end{bmatrix}$ using the low-rank approximation method stated in the lecture on Real Spectral Theorem.

- $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 2 - 2i \\ 0 & 2 + 2i & 2 \end{bmatrix}$
- $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 - i \\ 0 & 1 + i & 0 \end{bmatrix}$
- $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0.5 & 0.5 - 0.5i \\ 0 & 0.5 + 0.5i & 0 \end{bmatrix}$
- $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1.3333 & 0.6667 - 0.6667i \\ 0 & 0.6667 + 0.6667i & 0.6667 \end{bmatrix}$

No, the answer is incorrect.
Score: 0

Accepted Answers:
$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1.3333 & 0.6667 - 0.6667i \\ 0 & 0.6667 + 0.6667i & 0.6667 \end{bmatrix}$

12) Consider a normal operator $T$. Let $v_1$ and $v_2$ be eigenvectors of $T$ with eigenvalues $\lambda_1$ and $\lambda_2$, respectively.
Select the correct statement(s).

- $v_1$ is an eigenvector of $T^\ast$ with eigenvalue $\overline{\lambda_1}$.
- $2v_2$ is an eigenvector of $T^\ast$ with eigenvalue $2\overline{\lambda_2}$.
- $(v_1 + v_2)$ is an eigenvector of $T^\ast$ with eigenvalue $(\overline{\lambda_1} + \overline{\lambda_2})$.
- $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{R}$.

No, the answer is incorrect.
Score: 0
Accepted Answers:

$v_1$ is an eigenvector of $T^*$ with eigenvalue $\lambda_1$. 