1 Examples and Definitions

1a) Consider the dynamical system $\dot{\theta} = 1 + 2 \cos \theta$. Equating $f(\theta) = 0$, we can evaluate the fixed points of this system to be $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ for $0 \leq \theta \leq 2\pi$. In the intervals $\theta \in [0, \frac{2\pi}{3})$ and $\theta \in (\frac{4\pi}{3}, 2\pi]$, $f(\theta) > 0$. In the interval $\theta \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$, $f(\theta) \leq 0$. Hence, it is evident that $\theta = \frac{2\pi}{3}$ is a stable fixed point and $\theta = \frac{4\pi}{3}$ is an unstable fixed point. The phase portrait is given by Fig. 1.1.

![Phase portrait of $\dot{\theta} = 1 + 2 \cos \theta$.](image)

Figure 1.1: Phase portrait of $\dot{\theta} = 1 + 2 \cos \theta$.

1b) Consider the dynamical system $\dot{\theta} = \sin \theta + \cos \theta$. Equating $f(\theta) = 0$, we can evaluate the fixed points of this system to be $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ for $0 \leq \theta \leq 2\pi$. In the intervals $\theta \in [0, \frac{3\pi}{4})$ and $\theta \in (\frac{7\pi}{4}, 2\pi]$, $f(\theta) > 0$. In the interval $\theta \in [\frac{3\pi}{4}, \frac{7\pi}{4}]$, $f(\theta) \leq 0$. Hence, it is evident that $\theta = \frac{3\pi}{4}$ is a stable fixed point and $\theta = \frac{7\pi}{4}$ is an unstable fixed point. The phase portrait is given by Fig. 1.2.
2 Uniform Oscillator

1) The minute hand takes $T_1 = 1$ hour to finish one rotation and the hour hand takes $T_2 = 12$ hours. If $\theta_1$ represents the position of the minute hand, and $\theta_2$ that of the hour hand, we have

$$\dot{\theta}_1 = \frac{2\pi}{T_1},$$
$$\dot{\theta}_2 = \frac{2\pi}{T_2}. $$

If $\phi$ represents the phase difference between the two hands, we have

$$\phi = \theta_1 - \theta_2,$$

from which the rate of change of phase difference can be written as

$$\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2.$$

The two hands will overlap when the phase difference is $2\pi$. The time required for the phase difference to reach $2\pi$ can be computed using the rate of change of phase difference as

$$T_{\text{lap}} = \frac{2\pi}{\dot{\phi}} = \frac{2\pi}{\dot{\theta}_1 - \dot{\theta}_2} = \left( \frac{1}{T_1} - \frac{1}{T_2} \right)^{-1} = \frac{12}{11}.$$

Therefore, it takes $12/11$ hours for the two hands to overlap each other. If the hands first overlapped at 12:00, they would next overlap at 13:05 approximately.
3 Nonuniform Oscillator

1a) System in consideration is $\dot{\theta} = \mu \sin \theta - \sin 2\theta$. To obtain the fixed points, we equate $f(\theta)$ to zero. That is,

$$\mu \sin \theta - \sin 2\theta = 0$$

$$\implies \sin \theta (\mu - 2 \cos \theta) = 0$$

Thus, we obtain

$$\sin \theta = 0 \quad \mu - 2 \cos \theta = 0$$

$$\implies \theta = 0, \pi \quad \implies \cos \theta = \frac{\mu}{2}$$

Intuitively, we can see that $\mu = 0, -2, 2$ correspond to critical values of the parameter. We now plot the phase portraits to confirm this. The phase portraits are shown in Figure 3.1a-3.1e. The fixed points for each case are listed in Table 3.1.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\theta$</th>
<th>$\cos \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 2$</td>
<td>$-\pi, 0, \pi$</td>
<td>$\frac{\mu}{2}$</td>
</tr>
<tr>
<td>$0 &lt; \mu &lt; 2$</td>
<td>$-\pi, 0, \pi, \cos^{-1}(\mu/2)$</td>
<td></td>
</tr>
<tr>
<td>$\mu = 0$</td>
<td>$-\pi, -\pi/2, 0, \pi/2, \pi$</td>
<td></td>
</tr>
<tr>
<td>$-2 &lt; \mu &lt; 0$</td>
<td>$-\pi, 0, \pi, \cos^{-1}(\mu/2)$</td>
<td></td>
</tr>
<tr>
<td>$\leq -2$</td>
<td>$-\pi, 0, \pi$</td>
<td></td>
</tr>
</tbody>
</table>

It can be seen that fixed points are created and destroyed at the aforementioned critical values of $\mu$, and hence the system undergoes a saddle-node bifurcation.

4 Linear Systems

1) The given system is

$$\dot{x} = 4x - y,$$
$$\dot{y} = 2x + y.$$  

a) The above system can be re-written in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$
It is now in the form \( \dot{x} = Ax \), where
\[
\begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}.
\]

b) In order to find the characteristic polynomial, we need to evaluate
\[
\det(A - \lambda I).
\]
That is,
\[
\det \left( \begin{bmatrix} 4 - \lambda & -1 \\ 2 & 1 - \lambda \end{bmatrix} \right).
\]
This simplifies to
\[
(4 - \lambda) \times (1 - \lambda) + 2, \quad \lambda^2 - 5\lambda + 6.
\]
Therefore, \( f(\lambda) = \lambda^2 - 5\lambda + 6 \) is the required characteristic polynomial.

c) To find the eigenvalues, we need to solve the equation \( f(\lambda) = \lambda^2 - 5\lambda + 6 = 0 \). This yields \( \lambda_1 = 2 \) and \( \lambda_2 = 3 \) as the eigenvalues.

To obtain the eigenvector corresponding to an eigenvalue \( \lambda \), we need to solve the equation \( A\mathbf{x} = \lambda\mathbf{x} \). This computation yields the eigenvector \( \mathbf{v}_1 \) corresponding to the eigenvalue \( \lambda_1 = 2 \) and the eigenvector \( \mathbf{v}_2 \) corresponding to the eigenvalue \( \lambda_2 = 3 \), where
\[
\mathbf{v}_1 = \begin{bmatrix} a \\ 2a \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} b \\ b \end{bmatrix},
\]
where \( a, b \in \mathbb{R} \). For the particular value (for simplicity) of \( a = b = 1 \), we obtain
\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

d) For the matrix \( A \), we compute \( \Delta = \det(A) \) and \( \tau = \text{trace}(A) \). We obtain \( \Delta = 6 \) and \( \tau = 5 \). The quantity \( \tau^2 - 4\Delta \) evaluates to 1. Since \( \tau^2 - 4\Delta > 0 \), the origin is a node. Since \( \tau > 0 \), the origin is unstable. Hence, the origin is an unstable node for the given system.
Figure 3.1: Phase portraits for $\dot{\theta} = \mu \sin \theta - \sin 2\theta$, for various values of parameter $\mu$. 

(a) Regime: $\mu \geq 2$. Value: $\mu = 3$. 
(b) Regime: $0 < \mu < 2$. Value: $\mu = 1$. 
(c) Regime: $\mu = 0$. 
(d) Regime: $-2 < \mu < 0$. Value: $\mu = -1$. 
(e) Regime: $\mu \leq -2$. Value: $\mu = -3$. 

Figure 3.1: Phase portraits for $\dot{\theta} = \mu \sin \theta - \sin 2\theta$, for various values of parameter $\mu$. 

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