Introduction to Nonlinear Dynamics
Indian Institute of Technology Madras
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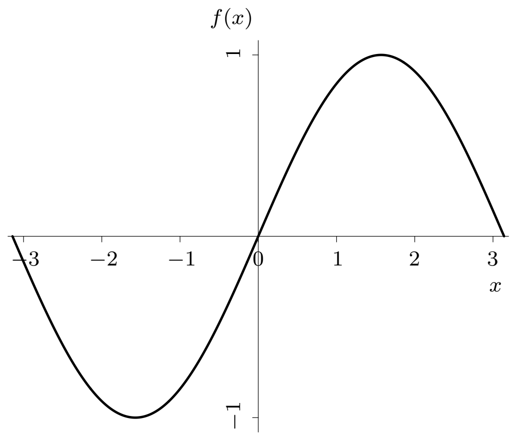
Solutions to Assignment 1

1 Geometric Intuition

1) [*Geometrical interpretation*]

a) $\dot{x} = \sin x$ is of the form $\dot{x} = f(x)$, with $f(x) = \sin x$. To obtain the fixed points, we graph f as a function of x and note down those x^* where $f(x^*) = 0$. Since $\sin x$ is periodic, we consider one period $(-\pi, \pi]$. As seen from the figure, in one period $(-\pi, \pi]$, $x^* = 0$ and $x^* = \pi$ are the two fixed points of $\dot{x} = \sin x$.



b) From the graph, it is clear that the velocity towards the right is maximum at $\tilde{x} = \pi/2$ in one period $(-\pi, \pi]$.

c) Given the system $\dot{x} = \sin x$, the acceleration is obtained by differentiating it with respect to *time*. Therefore, $\ddot{x} = \dot{x} \cos x$. Using the trigonometric identity $\sin(2x) = \sin x \cos x$, this can be further simplified to $\ddot{x} = \sin(2x)/2$.

d) Similar to part b), the maximum positive acceleration occurs at $\tilde{x} = \pi/4$ in one period $(-\pi/2, \pi/2]$.

2 Fixed points and stability

1) [*Analyse graphically*]
   Step 1: Find the fixed points.
   Step 2: Sketch the graph of $f(x)$ vs x.
   Step 3: Sketch the vector field on the real line, to find stability of fixed points. The flow is to the right when $f(x) > 0$, and to the left when $f(x) < 0$.

   a) $\dot{x} = 4x^2 - 16$
      To find the fixed points set $f(x) = 0$

$$f(x) = 4x^2 - 16 = 0$$
$$x^2 = 4$$
$$x = \pm 2.$$

Thus, there are two fixed points: $x = 2, -2$.
To sketch the graph of $f(x)$ vs x, we evaluate the value of $f(x)$ for a few values of x. The graph is as shown in Figure 2.1.
We then sketch the vector field on the real line.

x	0	1	-1	2	-2	3	-3	4	-4
$f(x)$	-16	-12	-12	0	0	20	20	48	48

(i) The flow is to the right when $x < -2$ and $x > 2$.

(ii) The flow is to the left when $x > -2$ and $x < 2$.

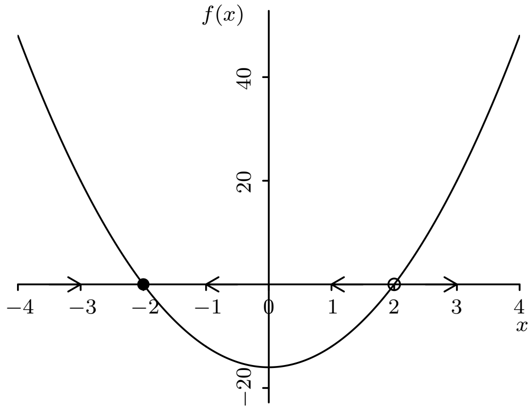


Figure 2.1: Plot of $f(x) = 4x^2 - 16$ vs x

Therefore, it is evident from the flow on the real line that -2 is a stable fixed point, and 2 is an unstable fixed point.

b) $\dot{x} = 1 - 2\cos x$
To find the fixed points set $f(x) = 0$

$$
\begin{aligned}
f(x) = 1 - 2\cos x &= 0 \\
2\cos x &= 1 \\
x &= \cos^{-1}\left(\frac{1}{2}\right) = \pm(2n+1)\frac{\pi}{3}
\end{aligned}
$$

PLot the graph of $f(x)$ vs x using some computing software such as MATLAB, or using the values of $\cos(x)$ from trigonometric tables. The graph is as shown in Figure 2.2. Then sketch the vector field on the real line.

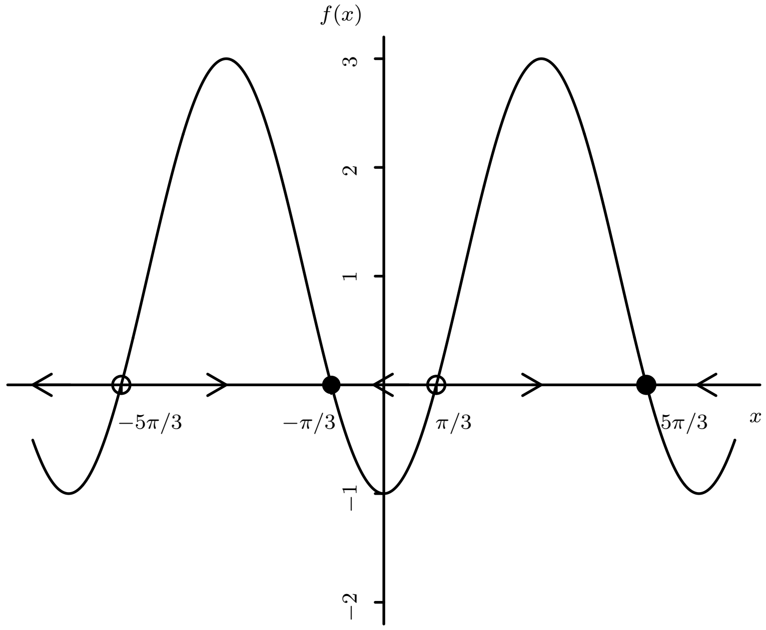


Figure 2.2: Plot of $f(x) = 1 - 2\cos x$ vs x

(i) The flow is to the right when $-5\pi/3 < x < -\pi/3$ and $\pi/3 < x < 5\pi/3$.

(ii) The flow is to the left when $-7\pi/3 < x < -5\pi/3$, $-\pi/3 < x < \pi/3$ and $5\pi/3 < x < 7\pi/3$.

We observe that $-5\pi/3$ and $\pi/3$ are unstable fixed points, whereas $-pi/3$ and $5\pi/3$ are stable fixed points.

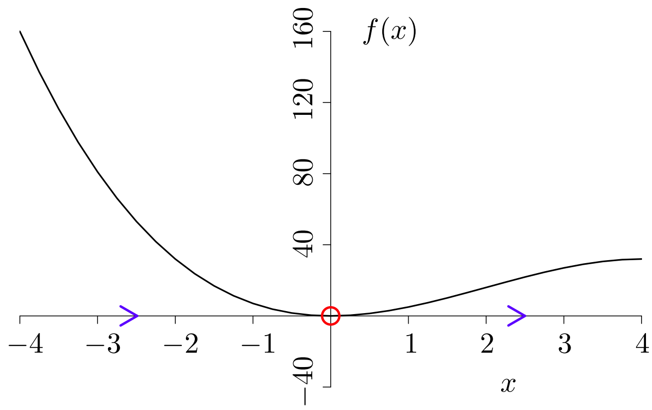


Figure 3.1: Plot of $f(x) = x^2(6-x)$ vs x

3 Linear stability analysis

1) Solution: Consider the dynamical system $\dot{x} = f(x)$
Step 1 : Let x^* be a fixed point of this system. Evaluate x^* by equating $f(x^*) = 0$.
Step 2 : Let $\eta(t) = x(t) - x^*$ be a small perturbation away from x^*. To see whether the perturbation grows or decays, we write a differential equation for η. Note that $\dot{\eta} = \dot{x}$. Now, using Taylor series expansion for $\dot{\eta} = f(x^* + \eta)$ and considering only linear terms, we get the following linear differential equation for η:

$$\dot{\eta} = \eta f^{'}(x^*)$$

- If $f^{'}(x^*) > 0$, $\eta(t)$ grows exponentially and hence x^* is unstable.
- If $f^{'}(x^*) < 0$, $\eta(t)$ decays exponentially and hence x^* is stable.

a) The fixed points of the system $\dot{x} = x(1-x)$ are $x^* = 0$ and $x^* = 1$ respectively. At $x^* = 0, f^{'}(x^*) = 1 > 0$. Hence, $x^* = 0$ is an unstable fixed point. At $x^* = 1$, $f^{'}(x^*) = -1 < 0$. Hence, $x^* = 1$ is a stable fixed point.

b) The fixed points of the system $\dot{x} = x^2(6-x)$ are $x^* = 0$ and $x^* = 6$ respectively. We see that, at $x^* = 0, f^{'}(x^*) = 0$, so we have to use graphical argument to comment about its stability. From the plot of $f(x)$ vs x which is given by Figure. 3.1, we can infer that on the right side of $x^* = 0$, $f(x) > 0$, so the direction of flow is away from x^* to the right. On the left side of x^*, $f(x) > 0$, so the direction of flow is towards $x^* = 0$ to the right. So, x^* is a half-stable fixed point. At $x^* = 6, f^{'}(x^*) = -36 < 0$. Hence, $x^* = 6$ is a stable fixed point.

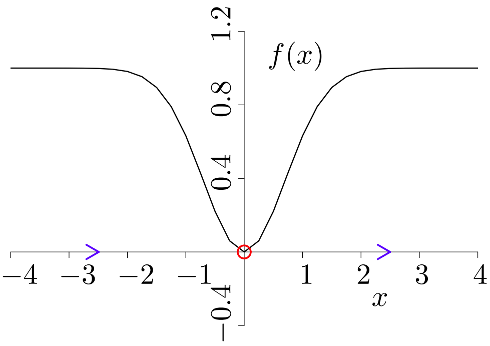


Figure 3.2: Plot of $f(x) = 1 - e^{-x^2}$ vs x

c) The fixed point of the system $\dot{x} = 1 - e^{-x^2}$ is $x^* = 0$. At this fixed point $f^{'}(x^*) = 0$. So, we have to use the graphical argument to comment on its stability. From the plot of $f(x)$ vs x as given in Figure. 3, we infer that at the right side of $x^* = 0$, $f(x) > 0$. Hence, the direction of flow is towards right away from $x^* = 0$. At the left side of $x^* = 0$, $f(x) > 0$. Hence, the direction of flow is right towards $x^* = 0$. Hence, $x^* = 0$ is an half-stable fixed point.