Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis

Solutions for Week 8 Assignment

April 21, 2017

NOTE: Wherever time-bandwidth product(TBWP) has been mentioned, unless stated otherwise, 'appropriate units' mean frequency being measured in Hz i.e. angular frequency in rad/s and time is measured in seconds.

1. Time and frequency variances are defined as, respectively:
   (a) $\frac{||tx(t)||_2^2}{||x(t)||_2^2}$, $\frac{||\frac{dx(t)}{dt}||_2^2}{||x(t)||_2^2}$
   (b) $\frac{||\frac{dx(t)}{dt}||_2^2}{||x(t)||_2^2}$, $\frac{||tx(t)||_2^2}{||x(t)||_2^2}$
   (c) $\frac{||\frac{dx(t)}{dt}||_2}{||x(t)||_2}$, $\frac{||tx(t)||_2}{||x(t)||_2}$
   (d) $\frac{||\frac{dx(t)}{dt}||_1^2}{||x(t)||_1^2}$, $\frac{||tx(t)||_1^2}{||x(t)||_1^2}$

   Ans: (a)

2. Which of the following is true for a Gaussian waveform, $exp(\frac{\gamma_o t^2}{2})$ where $\gamma_o < 0$:
   (a) Gaussian waveforms are used in Gaussian Mean Shift Keying(GMSK).
   (b) Gaussian waveforms cannot be realized in real time using some meaningful circuit.
   (c) Gaussian waveforms have a time-bandwidth product greater than that of a unit pulse function.
   (d) A causal system can have Gaussian waveform as its impulse response.

   Ans: (b)
   Explanation: Gaussian waveform as a matter of fact have the least time-bandwidth product. Gaussian waveforms have infinite length support - so can't be realized using a causal system. Same logic prevents its usage in real time application circuits. For the same reason, the waveforms used in GMSK use waveforms close to that of Gaussian, not exact Gaussian.

3. Consider a piece-wise function given by:

$$f(t) = \begin{cases} \frac{t^2}{2} & 0 \leq t \leq 1 \\ \frac{6t-2t^2-3}{2} & 1 \leq t \leq 2 \\ \frac{4-4(t-1)+(t-1)^2}{2} & 2 \leq t \leq 3 \\ 0 & otherwise \end{cases}$$

What can you say about time-bandwidth product(TBWP) of the above function?

(a) TBWP of the function is same as that of Gaussian waveform.

(b) TBWP of the above function is greater than that of a rectangular pulse function.

(c) TWBP of the above function is smaller than that of triangular pulse waveform.

(d) Nothing could be concluded about TBWP of the above waveform.

Ans: (c)
Explanation: Given function is actually a triple convolution of the rectangular pulse function, that is, result of convolving the pulse function with itself thrice. So, its time-bandwidth product will be smaller than that of both triangular and rectangular pulse function. Still, since it's not a Gaussian waveform, its not the optimal function and hence has time-bandwidth product greater than that of Gaussian waveform which is the optimal function.

Time-Bandwidth product(TBWP) is an important criteria while deciding which function to use in any multi-resolution analysis problem. Underlying phenomena responsible for it is the uncertainty principle that most of us are familiar with. The same uncertainty principle manifests itself in various forms across different disciplines. Since we know that a Gaussian waveform minimizes TBWP, it is likely that the same waveform can possibly give us Based on the information given above and your own understanding of the topic, answer the following 3 questions:(Unless stated otherwise, treat the units of frequency and time as mentioned above)

4. Time-bandwidth product is not independent of which of the following?

(a) Scaling and translations of dependent variable(s)

(b) Scaling and translations of independent variable(s)

(c) Bilinear transforms

(d) Fourier Transforms

Ans: (c)
Explanation: Time-bandwidth product is independent of the dependent, independent variables and Fourier Transforms. However, this invariance doesn't hold good for all bilinear transforms. For example, a bilinear transform mapping a rectangular pulse to a triangular one will change the time-bandwidth product.

Direction for next 4 Gabor Transform of a function $x(t)$ is given by $G_x(t, f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau) d\tau$. Answer the following 4 questions on the basis of this information and what you have learnt in the classes.

5. If $x(t) = e^{j2\pi f_0 t}$, then which of the following is true?

   (a) $|G_x(t, f)| = |G_x(t, f_0)|$ for all f.

   (b) $G_x(t, f_0) \leq G_x(t, f)$

   (c) $|G_x(t, f)| \leq G_x(t, f_0)$

   (d) None of the above

   Ans: (c)
   Explanation: $|G_x(t, f)| \leq \int_{-\infty}^{\infty} |e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} e^{j2\pi f_0 \tau}| d\tau = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} d\tau = G_x(t, f_0)$

6. To adjust the window size in Gabor Transform, we use an extra parameter σ such that the expression of Gabor Transform changes to:

   $G_x(t, f) = \int_{-\infty}^{\infty} e^{-\sigma\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau) d\tau$. Which of the following is correct:

   (a) Increasing σ results in larger effective window size

   (b) Decreasing σ results in larger effective window size

   (c) No change happens if change σ

   (d) None of the above

   Ans: (b)
   Explanation: Decreasing σ results in slower decay of the Gaussian and hence, larger effective window size.

7. What's the inverse Gabor Transform of the $G_x(t, f) = e^{-\pi t^2}$?

   (a) $x(t) = \delta(t)$

   (b) $x(t) = u(t)$, u(t): Unit Step function

   (c) $x(t) = rect(t)$, rect(t): Rectangular Pulse Function

   (d) None of the above

   Ans: (a)
   Explanation: We can see that option (a) is correct by plugging $x(t) = \delta(t)$ in the Gabor Transform equation given in the passage.

8. Which of the following is not a valid condition on the analysis function used in short-time fourier transform analysis?

   (a) Finite time domain support

(b) Finite time variance

(c) Finite frequency variance

(d) None of the above

Ans: (a)
Explanation: The two conditions on an analysis function are finite time and frequency variances.

9. For a chirp signal with linear $\Omega(t) = B \cdot t (B \neq 0)$(as mentioned in the class), we will obtain which of the following pattern in time-frequency plane? (Horizontal axis: Time axis and Vertical axis: Frequency Axis)

   (a) Tiles along a line which is not horizontal or vertical, get 'lit up'

   (b) Tiles along a horizontal line get 'lit up'

   (c) Tiles along a vertical line get 'lit up'

   (d) Depends on the window function we use

   Ans: (a)
   Explanation: Instantaneous frequency is a linear function of time with non-zero slope. So, we will see tiles centred on line of non-zero slope B being lit up. Also, functions whose $\mathcal{L}_2$ norm is not finite can have infinite extent in time and frequency domain. But functions which don't have finite $\mathcal{L}_2$ norm have undefined Fourier Transform.

10. Which of the following is an incorrect statement?

   (a) Every function occupies a finite area in time-frequency plane

   (b) Rectangular function can't be used as a window function in short time analysis

   (c) The so-called 'tile' in time-frequency plane has a minimum area of 1 in appropriate SI units.

   (d) None of the above

   Ans: (a)
   Explanation: Not all functions occupy finite area in time-frequency plane - only those which are square integrable do so. Rectangular window has infinite frequency variance and hence fails the criteria to be used a window function. Area of the tile is defined as $2\sigma_t \cdot 2\sigma_\Omega$ and hence using the lower bound on TBWP, we see that statement (c) is true.

11. Which of the following is an correct statement?

   (a) We can obtain Gaussian waveform as an impulse response of a Linear Time Invariant system

   (b) Time bandwidth product for $f(x) = \frac{(sin(Ax))^2}{(Bx)^2}$ where $A, B \neq 0$ is 0.3 in SI units

   (c) If TBWP of $f(x)$, where f(x) is a square integrable function, is 0.2, then TBWP of $f(x-1)$ is also 0.2.

(d) None of the above

Ans: (c)
Explanation: TBWP has a lower bound of 0.25 for all square integrable functions. So, (c) is not true. The function given in (b) is actually Fourier Transform of traingular pulse function which has TBWP as 0.3 and by the invariance of TBWP under Fourier Transform, the statement in (b) stands true. But for the (a), we can't obtain Gaussian waveform using a Linear Time Invariant system.

12. Which of the following is correct?

(a) In STFT, we have fixed tile size in time-frequency plane

(b) Using proper window function for short-term analysis, we get poor frequency resolution at higher frequencies and poor time resolution at lower frequencies.

(c) We can achieve a bounded TBWP using a non-compactly supported window function as well.

(d) All of the above

Ans: (d)
Explanation: In STFT, we use the window function without any scaling, so the tiles are of same size. Option (b) is the motive behind wavelet analysis. There is no constraint as compact support for TBWP to be finite. An example in case is the waveform $\frac{(sin(Ax))^2}{(Bx)^2}$ which is the fourier transform of Triangular pulse function and we know that TBWP is invariant under Fourier Transform and triangular function has finite TBWP.

13. In real life, we can't deal with infinite duration signals, so we generally limit the signal in time domain. Let $g(x)$ be a time-limited version of the standard Gaussian waveform $h(x) = e^{\frac{\gamma_0 x^2}{2}}$ where $Re(\gamma_0) < 0$, limited between $x = -t_0$ to $x = t_0$ ($t_0 > 0$ and is finite). Which of the following is true?

(a) As t_0 decreases, corresponding TBWP for $g(x)$ increases.

(b) $g(x)$ can't be obtained as an impulse response of a linear time invariant system

(c) h(x) can't be used as an analysis function in short time analysis if $Re(\gamma_0) > 0$.

(d) All of the above

Ans: (d)
Explanation: As we deviate from the ideal Gaussian waveform, the uncertainty product increases. So, (a) is true. An LTI system can't have a finite impulse response. SO, (b) is true as well. Finite time and frequency variance is needed for a function to be qualified as a window function, which is not the case with h(x).

14. Which of the following is correct in relation to CWT(continuous wavelet transform)?

(a) Wavelet function we use in CWT doesn't need to be normalized.

(b) Changing the scale parameter doesn't generate different filters in analysis.

(c) Changing the translation parameter generates different filters in analysis.

(d) None of the above

Ans: (c)
Explanation: Wavelet function we use, needs to be normalized. Changing the scale parameter results into filters with different bandwidths for example in case of Haar wavelet analysis (This is true in general as well). Change in translation parameter conserves the magnitude response but the phase response is different, in general. Thus, change in translation parameter generates different filters in analysis.