Assignment 2

Due on 6/2/18 11:59 PM ET

Unit 4 - Week 2:

1. Consider the set S = \{0, 1, 2\} with the operation \(\oplus\) defined by the following table:

<table>
<thead>
<tr>
<th>(a)</th>
<th>(\oplus)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

(a) Show that \(S, \oplus\) is a group under the operation \(\oplus\).

(b) Find the order of each element in \(S\).

(c) Determine if \(S\) is a field under the operation \(\oplus\).

(d) Find the inverse of each element in \(S\).

2. Suppose \(G = \{0, 1, 2\}\) is a group under the operation \(\oplus\) defined in question 1. Let \(H = \{0, 2\}\) be a subgroup of \(G\). Define the operation \(\circ\) on \(H\) by \(a \circ b = a + b \mod 3\) for all \(a, b \in H\). Show that \(H, \circ\) is a group under the operation \(\circ\).

3. Consider the group \(G = \mathbb{Z}/4\mathbb{Z}\) with the operation \(+\) where \(\mathbb{Z}/4\mathbb{Z}\) is the set of equivalence classes of integers modulo 4.

(a) Verify that \(G, +\) is a group.

(b) Find the inverse of \(2 + \mathbb{Z}/4\mathbb{Z}\) in \(G\).

(c) Consider the quotient group \(G/ \langle 2 + \mathbb{Z}/4\mathbb{Z}\rangle\). Is \(G/ \langle 2 + \mathbb{Z}/4\mathbb{Z}\rangle\) isomorphic to \(G\)?

(d) Determine the order of \(G/ \langle 2 + \mathbb{Z}/4\mathbb{Z}\rangle\).

4. Let \(G = \mathbb{Z}/6\mathbb{Z}\) be the group of integers modulo 6 under addition. Define a subgroup \(H = \{0, 3\}\) of \(G\).

(a) Find the left cosets of \(H\) in \(G\).

(b) Determine if \(G/ H\) is a group under the operation of coset multiplication.

(c) Show that \(G/ H\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\).

(d) Find the order of \(G/ H\).

5. Consider the group \(G = \mathbb{Z}/5\mathbb{Z}\) with the operation \(\times\) where \(\mathbb{Z}/5\mathbb{Z}\) is the set of equivalence classes of integers modulo 5.

(a) Verify that \(G, \times\) is a group.

(b) Find the inverse of \(3 + \mathbb{Z}/5\mathbb{Z}\) in \(G\).

(c) Consider the quotient group \(G/ \langle 3 + \mathbb{Z}/5\mathbb{Z}\rangle\). Is \(G/ \langle 3 + \mathbb{Z}/5\mathbb{Z}\rangle\) isomorphic to \(G\)?

(d) Determine the order of \(G/ \langle 3 + \mathbb{Z}/5\mathbb{Z}\rangle\).

6. Let \(G = \mathbb{Z}/12\mathbb{Z}\) be the group of integers modulo 12 under addition. Define a subgroup \(H = \{0, 6\}\) of \(G\).

(a) Find the left cosets of \(H\) in \(G\).

(b) Determine if \(G/ H\) is a group under the operation of coset multiplication.

(c) Show that \(G/ H\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\) or \(\mathbb{Z}/6\mathbb{Z}\), depending on additional structure.

(d) Find the order of \(G/ H\).

7. Let \(G = \mathbb{Z}/20\mathbb{Z}\) be the group of integers modulo 20 under addition. Define a subgroup \(H = \{0, 10\}\) of \(G\).

(a) Find the left cosets of \(H\) in \(G\).

(b) Determine if \(G/ H\) is a group under the operation of coset multiplication.

(c) Show that \(G/ H\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\) or \(\mathbb{Z}/10\mathbb{Z}\), depending on additional structure.

(d) Find the order of \(G/ H\).