Assignment 3

1. Given the following vector, find its magnitude and direction.

   \[ \mathbf{v} = (3, 4) \]

   - Magnitude: \( v = \sqrt{3^2 + 4^2} = 5 \)
   - Direction: \( \theta = \tan^{-1} \frac{4}{3} \approx 53.13^\circ \)

2. If \( \mathbf{a} \) and \( \mathbf{b} \) are two vectors, prove that \( (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \) for any vector \( \mathbf{c} \).

   - By definition: \( \mathbf{a} + \mathbf{b} \) is the sum of the vectors \( \mathbf{a} \) and \( \mathbf{b} \).
   - The dot product is distributive over vector addition:
     \[ (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \]

3. Find the dot product of the following vectors:

   - \( \mathbf{u} = (1, 2, 3) \) and \( \mathbf{v} = (4, 5, 6) \)
     \[ \mathbf{u} \cdot \mathbf{v} = (1)(4) + (2)(5) + (3)(6) = 32 \]

   - \( \mathbf{w} = (7, 8, 9) \) and \( \mathbf{z} = (10, 11, 12) \)
     \[ \mathbf{w} \cdot \mathbf{z} = (7)(10) + (8)(11) + (9)(12) = 274 \]

4. Suppose \( \mathbf{u} \) and \( \mathbf{v} \) are two vectors in \( \mathbb{R}^3 \) and \( \mathbf{u} = \mathbf{v} \).

   - Prove that \( \mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} \) for any vector \( \mathbf{w} \).

   - Since \( \mathbf{u} = \mathbf{v} \), then \( \mathbf{u} \cdot \mathbf{w} = (\mathbf{v})(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w} \).

5. A force \( \mathbf{F} = (3, 4, 0) \) N is applied to a 5 kg mass to produce an acceleration of 2 m/s^2. Find the work done by the force.

   - By definition of work: \( W = \mathbf{F} \cdot \mathbf{a} \)
   - Where \( \mathbf{a} = (2, 0, 0) \)
   - \[ W = (3)(2) + (4)(0) + (0)(0) = 6 \text{ J} \]

6. If \( \alpha, \beta, \gamma \) are the angles between \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) respectively, prove that \( \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \alpha \).

   - By definition: \( \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta \)
   - \( \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \alpha \) if and only if \( \theta = \alpha \).

7. A horizontal force of 20 N is applied to a 5 kg mass at an angle of 30° above the horizontal. Find the work done.

   - Using component form: \( \mathbf{F} = (20 \cos 30°, 20 \sin 30°) \) N
   - \[ W = (20 \cos 30°)(5) + (20 \sin 30°)(0) = 100 \sin 30° = 50 \text{ J} \]

8. If \( \mathbf{a} \) and \( \mathbf{b} \) are two vectors, prove that \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \).

   - By definition: \( \mathbf{a} \times \mathbf{b} = \mathbf{c} \) where \( \mathbf{c} \) is orthogonal to both \( \mathbf{a} \) and \( \mathbf{b} \) and satisfies the right-hand rule.
   - Since \( \mathbf{c} \) is orthogonal to \( \mathbf{a} \) and \( \mathbf{b} \), \( -\mathbf{c} \) will also be orthogonal to both vectors.
   - Thus, \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \).

9. A mass of 2 kg is hanging from a stiff spring attached to a horizontal bar. The spring constant is 10 N/m and the mass is subjected to a horizontal force of 10 N. Find the work done.

   - Using Hooke's Law: \( F = kx \)
   - \[ W = \int_{x_1}^{x_2} F \, dx = \int_{0}^{\frac{10}{10}} 10 \, dx = 10 \text{ J} \]

10. Two particles of masses 3 kg and 2 kg, respectively, are moving in a circular path of radius 4 m. If the particles are moving at an angular velocity of 2 rad/s, find the angular momentum of each particle.

    - Angular momentum for a particle of mass \( m \) and radius \( r \) moving at angular velocity \( \omega \) is \( L = mvr \).
    - For the 3 kg particle: \( L_1 = 3 \times 4 \times 2 = 24 \) kg·m^2/s
    - For the 2 kg particle: \( L_2 = 2 \times 4 \times 2 = 16 \) kg·m^2/s